Seminar Paper No. 37

OPTIMAL PRODUCTION AND GROWTH
POLICIES OF THE COMPETITIVE FIRM

by

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April 1974

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OPTIMAL PRODUCTION AND GROWTH POLICIES OF THE COMPETITIVE FIRM

I. Introduction

This paper deals with the theory of the growing firm under perfect competition on the factor and product markets. There seems to be in this field an open cleavage between the "verbal" literature (with Penrose [17] and Marris [13] as main originators) on the one hand, and more formalized micro theory on the other. While the first has suggested several interesting hypotheses on the dynamics of growing firms and industries, the second has in general not responded to this challenge with the necessary step out of the comparative static framework. The present paper makes a moderate attempt to bridge this cleavage by applying the tools of modern economic dynamics to a problem originally posed by the "verbal" tradition. This is the problem of behavior over time of a firm which has large "productive opportunities" (Penrose), but which is constrained by its present size in the exploitation of these opportunities.

The paper is organized as follows: after a short overview of earlier work in this field, Section II is built around a discussion on the nature of adjustment costs in the theory of the firm. The conclusion is that expansion of the firm is always costly by the very nature of the firm as an allocational device. On this basis a one-factor dynamic model of the firm is presented in Section III. In Section IV the properties of this model are studied.

*Helpful suggestions on construction and presentation of the model have been received from William A. Brock, David Cass, and Karl Shell. Earlier drafts of the paper have been presented at Seminars at the Institute for International Economic Studies with C.H. Siven as discussant and at the Stockholm School of Economics with Lars Svensson as discussant. My colleagues at the Institute for International Economic Studies have contributed with helpful comments. I would in particular like to mention Staffan Viotti, who has patiently and penetratingly commented on all parts of the paper over its entire "growth path". To all these I express my sincere gratitude.
and discussed. Section V opens with a discussion of the general problem of observing effects of parameter changes in dynamic structures. The comparative dynamics of this particular model is then studied, and short-run effects are compared with long-run effects. Section VI concludes the paper with a discussion of the applicability of the model and some of its implications.

II. Adjustment Costs in Team Production

Microeconomic theorizing in the field of economic dynamics can be said to have evolved so far around two main trends. The first is the analysis of firms and industries in steady-state growth (Baumol [3], Marris [13], etc.). The models used in these instances are built on the two assumptions of constant returns to scale and increasing costs to the rate of expansion. Therefore no stationary equilibrium exists, but only an equilibrium growth rate.

The second main trend is the study of the optimal adjustment of factors of production which deviate from the desired stocks. In this case a stationary equilibrium can be defined, but it is not optimal to attain it immediately because there are increasing costs to the rate of adjustment. Models of this type have been used to analyze optimal adjustment of capital stock (Eisner and Strotz [5], Gould [7], Lucas [12], Treadway [24]), the stock of knowledge (Lucas [11], Nordhaus [16]), the stock of goodwill (Gould [8]) and several other adjustment problems. One general feature of all these models is that they study the optimal adjustment of one particular factor of production, with cost of adjustment attached to it (the "adjustable" factor), while all other factors of production are freely variable. A typical result from the models is that the adjustment of the adjustable factor should be relatively large in the beginning of an optimal policy and then taper off, so that the optimal stock is approached asymptotically. A firm with a small capital stock should start out with a heavy investment program, a firm with little product goodwill should launch a big advertising campaign, etc.
But now, if a firm is small in all respects, can it follow all the optimal policies simultaneously? Can a firm make a "jump" by e.g. doubling within a short interval of time – its staff, capital stock, goodwill, technical knowledge, and so on? In short, how fast can (or should) a firm grow?

The answer to these questions depends on the type of adjustment problems that confront the firm. The economic factor which limits the rate of growth of the firm has often been described as external costs of adjustment, in the form of rising supply curves for factors of production. But this, of course, is a special case which does not apply to the perfectly competitive firm. In this paper we argue that internal costs of adjustment is the ultimate factor-limiting corporate growth, and that such costs are an inherent characteristic of the firm. In order to clarify these points, we must, in turn, enter a brief discussion of the nature of the firm.

One characteristic for coordination of factors of production in a firm is that the individual factors are not remunerated in direct relation to the marginal value product of their contributions. Such a direct relation would be a prerequisite for market coordination of productive efforts. One reason for the abolition of this direct relation was suggested by Coase [4] in his seminal paper on the nature of the firm. Coase pointed out the multiplicity of contracts to be made with the owner of each factor of production for all different contributions that could be needed, and the cost of negotiating and concluding all these contracts. Instead, the firm concludes with each (owner of a) factor of production one contract stating only "the limits to what the persons supplying the commodity or service is expected to do". The details can more cheaply be handled by central decision, organization and control than by market mechanisms.

Coase's point was enhanced and enlarged in a recent paper by Alchian and Demsetz [1]. An important factor for the existence of firms is – according to them – the non-separability of a team production function. Non-separability means that variations in
the level of input of one factor of production in a team does not only affect the marginal product of that factor but the marginal product of all other factors in the team. This property of the production function has two different implications for the organization of production. First, the number of contracts to be concluded under market organization is increased manyfold for every change in the level of output, because not only does the buyer of the final output have to bargain separately with each owner of a factor of production, but also these owners will have to bargain with each other, as the willingness of one to increase his level of input will depend on whether the others increase their levels of input as well. This tends to make market organization even more costly relative to non-market coordination. Secondly, the marginal product of the team due to a variation in the level of input of one particular factor of production (the marginal product of that factor in team production) may be very difficult to ascertain empirically. Therefore the rewards offered to a factor of production (by the final buyer or by the other factors in the team) might be completely out of line with its marginal product and hence preclude an efficient market allocation. And perhaps more important for our purposes - even if all involved (owners of) factors of production were completely altruistic (working for the benefit of the team only) they would not be able to make rational individual decisions on how, when and where to make their contributions to production, because they would not be able to observe what part of the outcome which was to be attributed to their effort.

Once it has been recognized that it is neither economical nor, perhaps, possible to reward a member of a team according to his marginal contribution, it is easily understood that other methods than economic incentives must be relied upon for coordination of production inside the firm. The type of coordination problem which confronts the firm has been studied in the economic theory of teams\(^1\) (see e.g. \([14]\))\(^1\). Generally speaking, the organization

\(^1\) The theory of teams deals with organizations in which the members have only common interests. In an organization like the firm, where members primarily further their own interests, there is - apart from the organization problem - an implementation problem in trying to make the members follow the prescribed rules. The second type of problem - the problem of shirking - has been extensively treated by Alchian and Demsetz \([1]\). Not having direct relevance for the phenomena treated in this paper, those problems are disregarded here.
of the firm must be built around one information structure, translating states of the world into information variables, and one decision structure, translating information variables into actions of the team members. The distinctive feature of the firm is - as we have seen - that individual members of the team cannot rationally turn observations on the state of the world into actions which further the interests of the team. Therefore, information must be processed by some central decision-making unit and then communicated in processed form (mainly in the form of direct instructions for action) to the members of the team. The information received by members from the central agent may be e.g. "bring unit x from department y" or "suggest market strategies for commodity z". The typical thing about such messages is that they are incomprehensible to an individual not belonging to the team. Not until a potential member of the team has been taught how to decode and react to different messages (not until a decision function has been imposed upon him) can he be an effective member of the team. This is in sharp contrast to the market, where observations on the state of the world can be directly turned into actions by a decision-making process involving the individual's utility function only.

Thus we have seen that there is a necessary element of learning in the team coordination of factors of production which characterizes the firm. The marginal contribution of a potential member of a team is zero until that member has been taught how to transform team-specific information into actions. Therefore, learning in this sense must be prior to production. It cannot be viewed as a joint output from the production process or from changes in the level of production. Rather, learning should be regarded as an investment in firm-specific knowledge. It transforms the "raw" factors of production available on the factor market into "effective" factors of production, i.e. team members with predictable responses to highly codified stimuli. And like all types of investment also this one is resource-consuming. Unlike other types of investment

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2) For this approach, see the seminal paper by Arrow [2], a classificatory paper by Fellner [6], and a later contribution by Rosen [19].
activities, however, the resources used in this one cannot be purchased on markets, but must be taken from within the firm. Only factors with firm-specific knowledge can transmit such knowledge to other factors. Or, to quote Penrose ([17], pp. 45-46), "... the capacities of the existing managerial personnel of the firm necessarily set a limit to the expansion of that firm in any given period of time, for it is self-evident that such management cannot be hired in the market-place."

For our purposes the firm can be conceptually divided into two departments: one production department, where team production is carried out, and one team formation department, where "raw" factors of production are transformed into effective team members. The two departments will compete for the available stock of members of the team. The most natural way to think about it may be to identify the team formation department with the empirically observable personnel department of a firm, where new labor is recruited, taught and trained. The firm is expanding at a maximum rate when all team members work full time in the personnel department. The cost of expansion is the value of output that could have been produced by these team members (either by direct participation or indirectly by their contributions to the well-functioning of the team), had they not been busy employing new labor.

The scope of the conceptual division of the firm into two departments is much wider than indicated by this example, however. Normally, every team member can be considered as belonging partly to the production department and partly to the team formation department. Work in one department is performed at the expense of work in the other. Examples can be found at all hierarchical levels of the firm. A managing director engaged in an expansion program will devote less time to the efficiency of the current operations of the firm. An engineer absorbed in an investment project will not be available for giving advice on a machine breakdown. When the finance department is busy with a new issue it will keep less track of the current cash-flow. When a worker instructs a new colleague, his machine stands idle for some time.
This, then, provides an answer to the question about the nature of adjustment costs, when a growing firm is considered. The firm is characterized by team production. Team formation is an investment, which requires the services of team members as an input. Therefore adjustment costs will always be present in the expansion of the firm. They will be present even if the firm is operating on perfectly competitive product and factor markets, because they are internal costs of adjustment. The argument can be condensed into three basic propositions, which are fundamental to the analysis of this paper:

1) Expansion of a productive team like the firm requires the input of resources already belonging to the team;

2) The rate of expansion is therefore limited by the amount of resources available in the firm at any point in time;

3) The cost of expansion of the firm is the opportunity cost of these resources in their best alternative use inside the firm.

From these propositions it follows directly that it is an economic problem of the firm to strike a balance between the rate of expansion and its current activities. This is the problem to be investigated in the present paper.

III. A One-Factor Model of the Growing Firm

In order to highlight the problems discussed in Section II the model of the firm set out in this section is simplified as much as possible in all respects which do not have a direct bearing on those particular problems.

Along the lines sketched in Section II consider a firm that is (conceptually) divided into two departments, one production department and one team formation department. The total stock of (homogeneous) factors of production in the firm is \( x \). For the sake of convenience \( x \) may be conceived of as man-hours. The number of man-hours going into the team formation department is \( y \), and so the number of man-hours left for the production department is \( x-y \).
The homogeneous output of the production department, \( q \), is sold on a perfectly competitive market at the constant price of 1 (i.e. the price of output is the numéraire). Man-hours are purchased on a likewise perfectly competitive market at price \( w \) (in terms of the numéraire). The production function of the production department is

\[
q = F(x - y)
\]  

assumed to be twice continuously differentiable with properties

\[
F(0) = 0 \quad F' > 0 \quad F'' < 0.
\]

We furthermore impose on \( F( ) \) the "Inada conditions", stating that

\[
F'(x - y) \rightarrow 0 \quad \text{when} \quad (x - y) \rightarrow \infty
\]

\[
F'(x - y) \rightarrow \infty \quad \text{when} \quad (x - y) \rightarrow 0.
\]

Turning next to the team formation department, the number of man-hours available here is – as mentioned – \( y \). "Output" of this department is a gross flow of new team members available to the firm (for use in either department). On the other hand, team members on the average quit the firm after having worked there for a certain period of time, so that in the absence of any team formation activities the available stock of effective labor will be subject to a certain rate of depreciation. The "production function" of the team formation department can now be written

\[
\dot{x} + \gamma x = \alpha y
\]  

where \( \dot{x} \) denotes the time derivative of \( x \), \( \gamma \) is the (proportional) rate of depreciation of the stock of effective labor, and \( \alpha \) is the marginal productivity of labor in team formation. For simplicity \( \alpha \) and \( \gamma \) are taken to be parameters in the present model.\(^3\)

---

3) With respect to \( \alpha \), falling marginal productivity in team formation (\( \dot{x} + \gamma x = g(y), \ g' > 0, \ g'' < 0 \)) is a feasible alternative (cf. models by Lucas [11] and Nordhaus [16], but to this model it adds nothing of substance. Increasing marginal productivity in team formation (Contd. on p. 10)
Following the discussion in Section II we now impose the following constraint on the firm:

$$x - y \geq 0,$$

(3)

which says that the firm can never use more resources for team formation than the total number of man-hours already employed. This is the basic dynamic restriction, because it excludes all possibilities of making "jumps" by pivins, together with (2), a maximum growth rate of the firm:

$$\dot{x} \leq \alpha - \gamma$$

or

$$\frac{\dot{x}}{x} \leq \alpha - \gamma$$

It is assumed that the firm can grow, i.e. that $\alpha > \gamma$, and indeed that $\alpha > (\rho + 2\gamma)$, where $\rho$ is the rate of interest.

It should be noted that while (3) constrains the flow of labor into the firm, it does not constrain the flow of labor between the two departments inside the firm. Labor, once turned into effective members, is assumed to be freely and instantaneously transferable between the departments. This requires that the information and decision structure of the firm is common to both departments so that there is no specialization of labor in either production or team formation activities. Qualifications of this clearly unrealistic assumption are discussed in the concluding section.

In the static theory of the firm the problem of the entrepreneur

3) (Contd. from p. 9) formation ($g'' > 0$), on the other hand, might give considerably different results (cf. Rothschild [20]), but this possibility is not investigated in this paper. With respect to the quit rate, $\gamma$, it could be made a (negative) function of the wage rate $v$ in relation to the average wage rate on the relevant labor market, but - the wage rate being parametric in this model - that approach is not pursued here (see however Salop [22]).
is to decide on the optimal stock of labor input. Now, with the flow of labor into the firm constrained by the capacity of the team formation department, his problem will be different. At any single moment in time the entrepreneur has to decide upon how much labor should be devoted to current production and how much to team formation activities. If the marginal value product of labor is higher than its marginal cost it will obviously pay to expand the firm. But there is a cost of expansion in terms of current production foregone, and the concavity of the production function makes the marginal opportunity cost rise with the rate of expansion.

The objective of the entrepreneur is assumed to be the maximization of the present value of the firm. Future profits are discounted at interest rate ρ. Also, at this rate of interest the supply of credit is assumed to be infinitely elastic.

IV. Optimal Production and Growth Policies

We are now in a position to collect our assumptions and present the entire model. With an infinite planning horizon the complete set of equations will be

\[
\begin{align*}
\max_{\mathcal{O}} V &= \int_{0}^{\infty} e^{-\rho t} [F(x - y) - wx] \, dt \\
\text{subject to} \quad & \begin{array}{l}
x = \alpha y - \gamma x \\
x - y \geq 0 \\
x, y \geq 0
\end{array}
\end{align*}
\]

This is a control problem in one state \(x\) and one control \(y\) variable where the time interval is unbounded. In our case, the number of man-hours used for team formation is the control variable and what we want to find is the optimal control, i.e. the time path of the control variable which maximizes the criterion function. This path then gives the behavior over time of the complete system.

In order to find the optimal control we set up the Hamiltonian of the system
\[ H = e^{-\rho t}[ F(x - y) - \omega x + \gamma \hat{p} \omega y - \gamma x + \hat{q}(x - y)] \] (6)

Because of the concavity of the Hamiltonian and the linearity of the constraints we know that the necessary conditions together with the transversality conditions are also sufficient.

**Necessary conditions, according to the Maximum Principle,** are

\[ \dot{\hat{p}} = \omega - F'(x - y) + (\rho + \gamma) \hat{p} - \hat{q} \] (7)

\[ \dot{\hat{\rho}} = (F'(x - y) + \hat{q})/\alpha \] (8)

\[ \hat{q} \geq 0, \quad \hat{q}(x - y) = 0 \] (9)

The transversality conditions for this problem are:

\[ \lim_{t \to +\infty} e^{-\rho t} \hat{p} = 0 \quad \lim_{t \to +\infty} e^{-\rho t} \hat{p} = 0 \] (10)

The new variable \( \hat{p} \) introduced in the Hamiltonian is a dynamic analogue of the Lagrangean multiplier called the auxiliary variable. Like the Lagrangean multiplier it can be interpreted as a price; specifically it fulfills the role of connecting stocks and flows of the system by being a shadow price of the stock in terms of the flow.

The variable \( \hat{q} \) in (6)-(9) is a Kuhn-Tucker multiplier, which as we can see from (9) takes on a non-zero (positive) value only when \( y = x \), i.e. in a boundary solution, when the firm is constrained by equality in condition (3). Because of the Inada conditions, however, the marginal opportunity cost of expansion goes to infinity as \( y \) approaches \( x \). Therefore the optimal solution will not be boundary at any point, so that \( \hat{q} = 0 \) always, and hence this

---

4) For derivations see Appendix I.
multiplier can be skipped.\(^5\)

The necessary conditions will now be:

I \( \dot{\alpha} = \dot{w} - F'(x-y) + (\rho + \gamma)\dot{\nu} \) \hspace{1cm} (7:a)

II \( \dot{\nu} = F'(x-y)/\alpha \) \hspace{1cm} (8:a)

Together with the original equation of motion

III \( \dot{x} = \alpha y - \gamma x \) \hspace{1cm} (2)

this constitutes a self-contained (autonomous) system, which is apt for qualitative solution.

Before solving the system, however, it might be useful to interpret the conditions derived. (7:a) can be written

\[
\dot{\nu} = \frac{F'(x-y) - w + \dot{w}}{\rho + \gamma}
\]

In this expression the numerator (the marginal product of labor minus its marginal cost plus capital gain or loss on labor) is the net flow of income of the firm arising from adding an extra man-hour

\(^5\) The optimal solution when \( F(\cdot) \) is quadratic (so that \( \dot{q} \) cannot be skipped) is partly boundary. This solution is discussed at the end of Appendix II.
to the team. When the stream of net incomes is discounted by the
relevant discount rate \( \rho + \gamma \) we get the capital value to the
firm of adding an extra man-hour to the team. This is \( \hat{\gamma} \). But \( \hat{\gamma} \)
also appears in condition \( (3:a) \). Here it should be noted that
\( 1/\alpha \) is the marginal requirement \( \gamma \) for increasing the flow of new
team members into the firm. \( F'(x - y) \) is the opportunity cost of
increasing \( y \). Hence, by \( (3:a) \), \( \hat{\gamma} \) is also the marginal cost of
expanding the team. Therefore, what is stated by the necessary
conditions is that \( y \) (the size of the team formation department)
should at all points in time be chosen so that the marginal pre-
sent value of additional labor in the team is equal to the marginal
cost of adding it to the team.

**Properties of the stationary equilibrium**

It is natural to start by studying the stationary solution of the
system. This solution is given by the two conditions \( \hat{\gamma} = 0 \) and
\( \hat{x} = 0 \). By the second condition we get from (2)

\[
\gamma^* \bigg|_{x=0} = \frac{(\gamma/\alpha)}{x}
\]  

Substituting (11) and \( (3:a) \) into \( (7:a) \) and rearranging we get

\[
\hat{\gamma} \bigg|_{x=0} = w - F'[x(1 - \gamma/\alpha)] \left[ 1 - (\rho + \gamma)/\alpha \right] \quad (7:aa)
\]

Applying the stationary condition \( \hat{\gamma} = 0 \), we know from our assump-
tions on \( F(\ ) \) that there is a unique solution \( x = x^* \) to this equa-
tion. \( x^* \) is hence uniquely determined by

\[
F'[x^*(1 - \gamma/\alpha)] = \frac{w}{1 - \rho + \gamma/\alpha} \quad (12)
\]

This, finally, supplies us with a stationary equilibrium value of
\( \hat{\gamma} \) from \( (3:a) \):

\[
\hat{\gamma}^* = F'[x^*(1 - \gamma/\alpha)]/\alpha \quad (13)
\]
The three values \( x^*, y^* = (\gamma/a)x^* \) and \( \theta^* \) together constitute the stationary equilibrium solution of the model.

When the firm has reached the size of \( x = x^* \), growth will come to a stop, and the number of man-hours employed will be constant over time. Some man-hours will still be devoted to team formation activities, but just enough to make up for normal turnover (as given by (11)). It might be worth noting that the stationary state is reached at a point where marginal productivity in current production, \( F'(x^* (1 - \gamma/a)) \), is not equal to the wage rate, \( w \). Rather, net team formation will stop on until the wage rate equals marginal productivity in current production minus the (opportunity) cost of keeping that labor stock constant. With the help of the concavity assumption on the production function (1), we can therefore draw the conclusion that the firm in stationary equilibrium will employ fewer workers in current production at each wage rate than predicted by a model where team formation is costless. This is a natural result as in our model "labor cost" is not only the wage cost but also the cost of forming the team. The relation with the standard "frictionless" model can be seen by studying equilibrium condition (12). When \( \alpha = \infty \), so that the costs of team formation go to zero, the standard condition

\[
F'(x^*) = w
\]

is obtained. On the other hand, letting \( \alpha + (\rho + \gamma) \) it is found that the stationary size of the firm goes to zero, so that a low marginal productivity in team formation has the result that the growth of the firm comes to a stop at an early stage. The extreme case arises for \( \alpha = (\rho + \gamma) \). In this case the costs of forming a team are prohibitive, and it will be more rewarding for labor to sell its services individually on the open market.

Properties of the optimal growth path

So far the system has been studied in stationary equilibrium, i.e.

6) Mathematically, of course, the firm never actually "reaches" \( x^* \) but only approaches it asymptotically. More rigorously, stationary equilibrium could be defined as a neighborhood of \( x^* \).
in a situation where employment and production do not change over time. Our main interest in this paper is not, however, in the study of such stationary situations, which we believe to be empirically largely irrelevant. Rather, we are interested in the dynamics of adjustment to this stationary point, and the behavior of firms that are in the process of adjustment.

There are many reasons why adjustment rather than stationariness seems to be the empirically pertinent *modus operandi* of the firm. In terms of our model the real wage rate, \( w \), is affected by cyclical and secular changes in both the nominal wage rate and the price of current output. The production function \( F(\cdot) \) is shifted by technological developments. What we observe in real life seems to be firms which are trying to catch up with their "productive opportunities", but these opportunities seem to be running always ahead of firms. Firms never reach their stationary states. We shall therefore concentrate our analysis on adjustments upwards (i.e. growth), and indeed, the model has been formulated to be particularly apt for the study of adjustments upwards rather than downwards.\(^7\) The next few sections will hence be devoted to the behavior of the firm on an equilibrium growth path.

Consider again the dynamic system of first-order necessary conditions:

\[
\begin{align*}
\text{I} & \quad \dot{p} = w - F'(x - y) + (\rho + \gamma)p \\
\text{II} & \quad \dot{p} = F'(x - y)/\alpha \\
\text{III} & \quad \dot{x} = \alpha y - \gamma x
\end{align*}
\]  

\[(7:a) \quad (8:a) \quad (2)\]

---

\(^7\) The model is not very realistic for adjustment downwards because employment is "irreversible": in decreasing the labor force the firm must wait for the "natural quit rate", \( \gamma \), to do the job.
The qualitative properties of this system can be studied in a phase diagram in the \((x, \dot{\nu})\)-phase. We first trace the loci of \((x, \dot{\nu})\) - combinations which give \(\ddot{\nu} = 0\) and \(\dot{x} = 0\), respectively. With a substitution of (8:a) into (7:a) we get
\[
\ddot{\nu} = \nu - (\alpha - \rho - \gamma)\dot{\nu}
\]  
(7:ab)

Therefore \(\ddot{\nu} = 0\) whenever \(\dot{\nu} = \nu/(\alpha - \rho - \gamma) = \nu^*\), regardless of the value of \(x\). The locus of \(\ddot{\nu} = 0\) is hence a horizontal line in the \((x, \nu)\)-plane (see Fig. 1). When \(\dot{\nu} > \nu^*\), \(\ddot{\nu} < 0\) and vice versa, as indicated by the small vertical arrows in the figure.

\[\nu^*\]

\[\nu\]

\[\nu_0\]

\[\nu_1\]

\[\nu_2\]

\[x_0\]

\[x^*\]

\[\dot{\nu} = 0\]

\[\dot{x} = 0\]

\[x\]

Figure 1
Next consider \( \dot{x} = 0 \). We know (from (11)) that this implies \( y = (\gamma/\alpha)x \). Substituting this into (8:a) gives

\[
\frac{\partial y}{\partial \dot{x}} \bigg| _{x=0} = F' \left[ x(1 - \gamma/\alpha) \right]/\alpha
\]

(15)

and differentiating with respect to \( x \)

\[
\frac{\partial y}{\partial x} \bigg| _{x=0} = \left[ 1 - \gamma/\alpha \right] F'' \left[ x(1 - \gamma/\alpha) \right]/\alpha < 0
\]

we find that the locus for \( \dot{x} = 0 \) in the \((x, \dot{y})\) plane is a negatively sloped curve, the curvature of which depends on \( F''(\cdot) \). As drawn in Fig. 1 the \( \dot{x} = 0 \) curve is consistent with the Inada conditions. Any point above the \( \dot{x} = 0 \) curve represents a \( \ddot{y} \) higher than indicated by (15) for given \( x \). As (from (8:a)) \( y \) is a positive function of \( \dot{y} \) when \( x \) is constant, we understand that \( \dot{x} > 0 \) for all points above the \( \dot{x} = 0 \) curve. The opposite holds for points below that curve, as indicated by the small horizontal arrows in Fig. 1.

The intersection of the \( \dot{y} = 0 \) and \( \dot{x} = 0 \) curves represents, of course, the stationary equilibrium \((x^*, \dot{y}^*)\) already analyzed. We are interested now in the optimal behavior of the firm in approaching this equilibrium. It is clearly seen that the paths from an arbitrary point in Fig. 1 which fulfill the necessary conditions do not in general approach equilibrium. We must find now which one of the infinite number of time paths fulfilling the necessary conditions actually approaches equilibrium.

Remember that \( \ddot{y} \) is a single-valued positive function of \( y \) for any given \( x \). Now let a firm be of the size \( x_0 < x^* \). At that low level of employment the marginal value product of labor is higher than labor cost, so the firm will want to hire more labor until it reaches \( x^* \). How much of the present labor force should be devoted to team formation? To find this, let the firm choose an arbitrary \( y \) corresponding to, say, \( \ddot{y}_1 < \ddot{y}_0 \). If the firm follows the necessary conditions (7:a), (8:a) and (2) from this starting point, then it will be on a finally contractive time path, which comes to an end at \((0, \ddot{y}^*)\), as indicated by the thin, curved arrow starting at \((x_0, \ddot{y}_0)\). Here,
obviously, too little labor was allocated to team formation. On
the other hand, let the firm choose \( y \) consistent with \( \hat{\nu}_2 > \hat{\nu}_0 \). In
this case the time path which fulfills the necessary conditions
leads to \((x^*, \hat{\nu}^*)\) because too much labor was allocated to team for-
modation from the beginning. This is indicated by the thin, curved
arrow starting at \((x_0, \hat{\nu}_0)\). Either of two such diverging paths
results from all \( \hat{\nu} \neq \hat{\nu}_0 \), so we are left with \( \hat{\nu}_0 \) as the only initial
choice for a time path which both fulfills the necessary conditions
and leads to a finite non-zero stationary equilibrium, i.e. the only
feasible optimal solution.\footnote{Mathematically, the diverging paths can be
discarded on the ground that they do not fulfill the transversality conditions
(10).} This unique feasible optimal time path
is indicated in Fig. 1 by the heavy, dashed arrows.

There is a unique feasible optimal \( \hat{\nu} \) (and hence \( y \)) for any initial
\( x \), as may be verified diagrammatically in Fig. 1. There is there-
fore a unique \( \varepsilon \)rowth policy which is optimal to the firm. It is
important to note here that what has been derived is a policy - or
program - rather than the actual behavior over time of the firm.
In the "initial position" \( x_0 \) the entrepreneur makes a complete
blueprint of the development of the firm up to \( x^* \). This blueprint
is based on his criterion function \( V_0 \) and (expectations held with)
certainty about all parameters of the model. With the form of
criterion function chosen, the entrepreneur will stick to the pro-
gram as originally worked out (provided parameters turn out as
expected) even if he is allowed to replan at a later stage of
development. If, however, his expectations with respect to para-
meters turn out to be erroneous, he will recalculate the \textit{entire}
\( \text{optimal program} \) and immediately switch over to the new program.
This situation is dealt with in Section V. In this section we
discuss the optimal policy as carried out when parameter values
remain constant (as expected) over the entire planning period.
What do we know about this optimal policy? A few qualitative
observations will be made here. Note again that we restrict our
attention to the growth path, i.e. to situations where the initial $x < x^\ast$.

From the analysis above we know that $\dot{x} > 0$ and $\ddot{y} < 0$ along the entire optimal growth path. This helps a bit because, by differentiating (8:a) with respect to time, we get

$$\dot{y} = \dot{x} - \frac{\ddot{y}}{F''(x - y)}$$

(16)

This means that $\dot{y}$ can never exceed $\dot{x}$, so that in an optimal growth policy $y/x$ is never increasing over time. Team formation labour will never increase as a proportion of total labor. This, in turn, implies that the absolute size of the production department will never decrease, so neither will output on an optimal growth path.

These arguments have some interesting implications for productivity measurement. The size of the team formation department will naturally be an important factor determining the observed productivity of the firm, as measured by $q/x$. As the firm grows and approaches its stationary equilibrium the team formation staff becomes a smaller and smaller proportion of total employment. This will be a counteracting force to the fall in productivity as the firm grows, which is implied by the shape of the production function: When the firm is small it will operate with a (relatively) small highly productive production department and a (relatively) large team formation department; when it grows bigger the proportion of production workers will increase, but on the other hand productivity in direct production will fall as a result of the increase in size. The total effect on average productivity over time is indeterminate in the absence of a more specified production function.

The absolute value of the second term on the RHS of (16) can be smaller as well as larger than $\dot{x}$, so that $\dot{y}$ may be either positive or negative. One possible development over time is shown in Fig. 2. In this case $y$ increases ($\dot{y} > 0$) up to size $x^\ast$ and thereafter.

---

9) This is a specific variant of a more general result from applications of control theory, which says that the control should be applied most heavily in the initial periods and decrease continually thereafter. (See e.g. Jacquemin and Thisse [9].)
decreases ($\dot{y} < 0$) until the stationary state is reached. On the path up to $x^o$ we may speak of size-constrained growth, because during this period (with $\dot{y} > 0$) the optimal policy deviates from the standard result that the absolute value of the control should be falling over the entire optimal path. As noted above (footnote 9), we have in our case a relative variant of the same rule, i.e. $y/x$ falling over the entire optimal path.

It may be instructive to see the described process also in the $(x,y)$-phase, as shown in Fig. 3. The feasible area in this phase is restricted to be below the $x = y$ line by condition (3). From (11) we know that the locus of all points, such that $\ddot{x} = 0$, is a straight line with positive slope $\gamma/\alpha$. Above this line $x$ is increasing and below it $x$ is decreasing as marked by the small horizontal arrows. Under certain (rather restrictive) assumptions with respect to higher derivatives of the production function, it can be demonstrated (see Appendix II) that the $\dot{y} = 0$ schedule has the shape.
Given to it in Fig. 3. In any case we know from the previous analysis that at one (and only one) point on the $\dot{x} = 0$-line is $\dot{y}$ also equal to zero. Let this point be $(x^*, y^*)$. We can now draw an example of an optimal path into this diagram as shown by the heavy, dashed arrow. In order to be consistent with the example given in Fig. 2, the optimal path is drawn with $y$ rising up to size $x^0$ and falling thereafter. The absolute size of the team formation department over the growth program can be explicitly studied in the diagram. Size-constrained growth takes place in that part of the optimal program which goes from the origin to $x^0$, when the absolute size of the team formation department is increasing.

V. Comparative Dynamics and the Instantaneous Effects of Parameter Changes

In comparative statics stationary equilibria before and after a parameter change are compared. When the optimal adjustment is distributed over time, as is the case in the model studied here, the observed effect of a parameter change will depend on whether
or not the firm is in stationary equilibrium when the observations before and after the change are made. As mentioned earlier, the effect of a parameter change (unexpected by definition) is to make the firm replan the entire optimal growth path. It will then instantaneously switch over to the new path. The effect of the parameter change on the adjustment path may be quite different from the effect on the stationary equilibrium. Therefore, the type of equilibrium conditions (dynamic or stationary) which are affected by the parameter change must be known before the effect can be determined. As will be clear from the analysis, this argument applies not only to the magnitude of the effect, but also to its direction.

With respect to the type of equilibrium conditions at the time of the observations, four different observation combinations are possible as given by Table 1. Comparative statics covers only case D in this table, while we take comparative dynamics to cover all cases A - D.

<table>
<thead>
<tr>
<th>Observation before parameter change</th>
<th>After parameter change</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Equilibrium condition</td>
</tr>
<tr>
<td></td>
<td>Optimal adjustment</td>
</tr>
<tr>
<td></td>
<td>Stationary equilibrium</td>
</tr>
<tr>
<td>Before parameter change</td>
<td></td>
</tr>
<tr>
<td>Optimal adjustment</td>
<td>A</td>
</tr>
<tr>
<td>Stationary equilibrium</td>
<td>C</td>
</tr>
</tbody>
</table>

As an illustration of the argument consider a variable \( z = z(\beta,t) \), where \( \beta \) is a parameter. See Figure 4! \( z = z(\beta_1,t) \) approaches stationary equilibrium \( z_1 \) over time. Then, at time \( t_3 \), \( \beta \) is changed from \( \beta_1 \) to \( \beta_2 \). \( z(\beta_2,t) \) now begins to approach a higher
stationary equilibrium $z^*_2$, but this adjustment starts from a lower current level of the variable. As the adjustment path is drawn it is clear that the timing of the "before" and "after" observations becomes a crucial issue in judging the effects of the parameter change. Possible cases are given in Table 2, with capital letters referring to Table 1.

**Table 2**

<table>
<thead>
<tr>
<th>Observation at $\beta = \beta_1$ at $\beta = \beta_2$</th>
<th>Case</th>
<th>$\Delta z / \Delta \beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$ to $t_4$</td>
<td>A</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>$t_1$ to $t_5$</td>
<td>B</td>
<td>$&gt; 0$</td>
</tr>
<tr>
<td>$t_2$ to $t_4$</td>
<td>C</td>
<td>$&lt; 0$</td>
</tr>
<tr>
<td>$t_2$ to $t_5$</td>
<td>D</td>
<td>$&gt; 0$</td>
</tr>
</tbody>
</table>

In the particular case illustrated by Figure 4 it is clear that - even though the comparative statics is unambiguous - the observed effect of the parameter change may go in either direction depending on when the observations are made. Such "perverse" cases do in fact occur in the model studied here, and it is therefore important to study both the "instantaneous" (at time $t_3$) and comparative static ($z^*_1 - z^*_2$) effects of a parameter change.
Method

For the comparative dynamics we apply a method which has been used earlier by Treadway [24, 25] and Gould [8]. Consider again Figure 1 (reproduced below as Figure 5). The equations for the \( \ddot{p} = 0 \) and \( \dot{x} = 0 \) loci in this figure can be written

\[
\ddot{p} \bigg|_{\ddot{p}=0} = w/(\alpha - \rho - \gamma) \quad (7:ac)
\]

\[
\ddot{p} \bigg|_{x=0} = F'[x(1 - \gamma/\alpha)]/\alpha \quad (15)
\]

A parameter change will result in a shift in either of the two curves (or in both). The new stationary solution is found at their new point of intersection, giving \( \ddot{p}^* \) and \( x^* \). Once these are established we can easily find \( y^* = (\gamma/\alpha)x^* \) (the optimal stationary team formation staff) and \( q^* = F(x^* - y^*) \) (the optimal stationary level of production). A comparison of these new stationary solutions with the original stationary equilibria provides us with the analogue of comparative static results of the model.
What about the "instantaneous" effects of parameter changes? As \( x \) can only be changed by the equation of motion \( \dot{x} = a_y - \gamma x \), there can be no instantaneous change in \( x \). Instantaneous effects are therefore restricted to the variables \( \bar{p} \), \( y \), and \( a \), with \( x \) given. In terms of Figure 4 these effects can therefore be observed as "vertical jumps" from the old optimal path (connected with the old stationary equilibrium) to the new optimal path (leading to the new stationary equilibrium). Conclusive answers on whether the jump is upwards or downwards can be given if the new path is everywhere above or below the old path. This, of course, depends both on the relative positions of the stationary equilibria and on the slopes of the optimal paths. The slope of the optimal path in the \((x, \bar{p})\)-phase is given by

\[
\frac{d\bar{p}}{dx} = \frac{\dot{\bar{p}}}{\dot{x}} = \frac{w - (a - \phi - \gamma)\bar{p}}{a_y - \gamma x}.
\]  

In some cases it may happen that the two optimal paths intersect, and so no general qualitative conclusions are possible. If they intersect, then the growth path leading to a stationary equilibrium above the old path must be less steep than that path in at least one point of intersection.

Differentiating (17) with respect to a parameter (e.g. \( \beta \)) shows whether a new optimal path becomes steeper (\( \partial(\dot{\bar{p}}/\dot{x})/\partial\beta < 0 \)) or less steep (\( \partial(\dot{\bar{p}}/\dot{x})/\partial\beta > 0 \)) in a potential point of intersection when the parameter changes. If the new stationary equilibrium is above the old path, and the slope of the new path is unambiguously steeper in any arbitrary point, then obviously the two paths cannot intersect. In such cases it is possible to tell whether a switch from the old path to a new one implies a jump upwards or downwards, irrespective of where on the old path the firm happens to be, and hence general qualitative statements can be made about the effects of parameter changes on a firm which is not in stationary equilibrium.

Before proceeding, it should be underlined that the comparative dynamic experiments to be performed below are all made under the
assumption of static expectations. This means that a parameter change is unexpected and the new parameter value is expected to last forever. The effect of a known future parameter change which lasts only for a limited period of time cannot be investigated by the present method. It can only be conjectured here (see Gould [7], pp. 52-54) that such a change would have a similar qualitative effect from the moment it became known until it was over.

A change in the real wage rate

The way the model has been formulated a rise in the real wage rate, \( w \), everything else held constant, can be used to represent not only a rise in the nominal wage rate with product price held constant, but also a fall in the price of current output (which is used as numéraire) with the nominal wage rate held constant. The present exercise will therefore provide not only a factor demand function, but also a supply function for current output.

Consider first the effect of an increase in the real wage on the stationary equilibrium. Differentiating (7:ac) and (15) with respect to \( w \) gives

\[
\begin{align*}
\frac{\partial  \tilde{p}}{\partial w} \bigg|_{x=0} &= 0 \quad \text{and} \quad \frac{\partial  \tilde{p}}{\partial w} \bigg|_{p=0} = \frac{1}{\alpha - \rho - \gamma} > 0
\end{align*}
\]

The effect is therefore an upward shift in the \( p=0 \)-schedule while the \( x=0 \)-schedule remains unaffected as shown in Figure 6. It is quite clear from the figure that an increase in the wage rate will decrease stationary equilibrium employment, as expected. The level of output in stationary equilibrium will, of course, also be lower, because \( q^* = F(x^*(1 - \gamma/a)), F' > 0. \)
As the $\dot{x}=0$-schedule is not affected by changes in the real wage rate, all stationary equilibria for different wage rates will be on that schedule. Therefore, from (7:ac), if the scale on the $\tilde{p}$-axis is multiplied by $(\alpha - \rho - \gamma)$, the $\dot{x}=0$ curve gives the stationary equilibrium relation between $w$ and $x$. Hence, the $\dot{x}=0$-curve can be interpreted as an "ordinary" stationary equilibrium factor demand curve for $x$. Also, as $w$ is the nominal wage rate divided by the price of current output, an upward movement along the $\tilde{p}$ axis can be taken to represent a decrease in output price with the nominal wage rate held constant. With this interpretation the $\dot{x}=0$-schedule becomes an "ordinary" (upward sloping) supply curve for current output.

The main purpose of this exercise in the comparison of stationary equilibria has been to show that the model is "well-behaved", i.e. that in stationary equilibrium it gives qualitatively the same results as a conventional comparative static model. We now move on to the study of the dynamics outside stationary equilibrium. The two stationary equilibria each have an optimal path (dashed curve) connected to them. The optimal path connected to $(x^*_1, \tilde{p}^*_1)$ is drawn in Figure 6 everywhere above the path leading to
(x₂*, p₂*). Two arguments show that this must be the case. First, it is clear that (x₂*, p₂*) is below the growth path leading to (x₁*, p₁*). Secondly, an optimal growth path associated with a higher w is less steep at any arbitrary point than one associated with a lower w, as can be verified by differentiating (17) with respect to w

\[ \frac{\partial(\bar{p}/\bar{x})}{\partial w} = \frac{1}{\alpha y - \gamma x} > 0. \]

So the new path will be below the old one for all x values, and they can never intersect.

Now, let a firm be on its optimal path to x₁*. Then - before the firm has reached the size of x₂* - the nominal wage rate is increased or the price of current output is decreased, everything else remaining unchanged. The entrepreneur - who had earlier been certain that the old nominal wage rate would remain throughout the growth program - now has to replan a new growth program on the basis of the new nominal wage rate. The new optimal path is the one leading to x₂*, and a present-value-maximizing entrepreneur will let the firm make a vertical jump onto the new path. The jump implies a sudden decrease in \( \bar{p} \) as indicated by the dotted curve in Figure 6. This decrease in \( \bar{p} \) reflects a lower valuation of marginal additions to the stock of effective labor, as the stream of net income from this labor is diminished by the rise in the real wage rate. The response of the firm is given by the necessary condition

\[ \bar{p} = F'(x-y)/\alpha \] (8:a)

i.e. the marginal cost of team formation should fall accordingly. With x and α given the only way to lower marginal team formation costs must be a decrease in the size of the team formation department, y (remember F'' < 0). This in turn has two effects: 1) the rate of growth of the firm (\( \dot{x} \)) is decreased, and 2) the rate of current production (q = F(x-y)) is increased. While the first of these "instantaneous" effects does not seem very controversial, the second certainly is, implying as it does a negatively sloped "instantaneous" supply curve for current output. This, however, is a natural result of the model as stated: when the price of current output rises (and - of course - the new price is expected
with certainty to persist indefinitely) it becomes profitable for
the firm to grow at a quicker pace, but this requires increased
resources for team formation and these resources then have to be
taken out of current production. Conversely, when product price
declines or the wage rate increases resources are transferred
from the team formation to the production department. The imme-
diate effect - of course - is an increase in current output.

We shall return later to the assumptions of the model which lead
to this perverse result.¹⁰

Changes in the marginal productivity in team formation and the
quit rate

Changes in \( \alpha \) and \( \gamma \) have the effect of shifting both the \( \dot{x}=0 \)-curve
and the \( \ddot{p}=0 \)-curve in the same direction. An increase in \( \alpha \) will
shift both curves downwards, whereas an increase in \( \gamma \) will move
both curves upwards. (The skeptical reader may verify this for
himself by differentiating (7:ac) and (15) with respect to \( \alpha \) and
\( \gamma \).)

As the two cases are very similar we shall here only investigate
a change in \( \alpha \). The effect of an increase in \( \alpha \) is shown in Fig. 7
below.

![Diagram](image)

Fig. 7

¹⁰ The effect has been observed earlier by Lucas [12] and Rubin [2].
The effect on $x^*$ is ambiguous and depends, among other things, on the shape of the production function. This result is not surprising as two counteracting forces are involved here. First, an increase in $\alpha$ lowers the (opportunity) cost of increasing the stock of team members which has a positive effect on size. Second, a rise in $\alpha$ makes for a smaller need for overhead labor, which has a negative effect on optimal stationary size. The net outcome— as mentioned—is ambiguous.

It cannot be ascertained in this case that the optimal paths do not intersect, and therefore no conclusive instantaneous effects can be established. This again is a result of the two opposing forces described above. Hence, we must leave this particular case with the contention that very little can be said about it. What has been said about an increase in $\alpha$ also applies qualitatively to a decrease in $\gamma$, and vice versa.

**A change in the rate of interest**

In the comparative static case a change in the rate of interest will affect production and factor input "in the long run", i.e. when the capital stock is variable, because the rate of interest enters into the cost of capital. In our one-factor model it seems natural that the stationary equilibrium stock of factor input should also be affected by a change in the rate of interest. Again, this effect can be found by differentiating (15) and (7:ac) with respect to $\rho$.

$$\frac{\delta \hat{p}}{\delta \rho} \bigg|_{x=0} = 0$$

$$\frac{\delta \hat{p}}{\delta \rho} \bigg|_{p=0} = \frac{w}{(\alpha - \rho - \gamma)^2} > 0$$

The upward shift in the $\hat{p}=0$-curve gives a case very similar to the one where the wage rate was changed. Indeed, we find again that the new optimal growth path is everywhere below the old one, because

$$\frac{\delta (\hat{n}/x)}{\delta \rho} = \frac{\hat{p}}{\alpha \gamma - \gamma x} > 0$$
The case can therefore be illustrated by Fig. 6 above. The new stationary equilibrium occurs, as expected, at a lower level of production and factor input. Behind this effect is an ordinary capital-theoretic insight. In our model the firm must weigh the stream of future net income arising from an extra team member against the present (opportunity) cost of turning him into a team member. Now, when the interest rate rises one will be willing to sacrifice less today for a given benefit in the future. Therefore the firm will stop net team formation at an earlier stage than before.

The instantaneous effect of an increase in the rate of interest is an increase in current output, as labor is moved over from the team formation to the production department. This effect, of course, is as "perverse" as the similar response to a change in the real wage rate. Another immediate result of the transfer of resources from the team formation to the production department is that the rate of growth of the firm will slow down. This effect of an increase in the rate of interest is - in contrast to the effect mentioned above - well known not only from capital theory but also from the more recent literature on the steady state theory of the growth of the firm (see e.g. Baumol [3]).

A tax on profits

The introduction of a profits tax in the present model requires a slight reformulation of the maximization problem of the firm. It is not immediately clear what should be meant by "profits". If we define it as total revenue minus total cost (which we may call "Profit 1"), the new formulation becomes

\[
\max_{\gamma} V_0 = \int_{\gamma}^{\gamma_0} e^{-\rho t} [F(x - \gamma) - wx](1 - \tau_1) dt
\]

subject to the same constraints as before. \(\tau_1\) is here the (proportional) tax rate on "Profit 1". In this case the new stationary conditions become

\[
\bar{p}|\hat{x}=0 = (1 - \tau_1)F'[x(1 - \gamma/a)]/a
\]  

(18)
and

\[ \frac{\delta \tilde{p}^2}{\delta \tau_1} \bigg|_{p=0} = \frac{(1 - \tau_1)w}{(\alpha - \rho - \gamma)}, \quad (19) \]

so an increase in the tax rate will give

\[ \frac{\delta \tilde{p}^2}{\delta \tau_1} \bigg|_{x=0} = \frac{F'[x(1 - \gamma/a)]}{\alpha} < 0 \quad (20) \]

and

\[ \frac{\delta \tilde{p}^2}{\delta \tau_1} \bigg|_{p=0} = -\frac{w}{\alpha - \rho - \gamma} < 0 \quad (21) \]

In stationary equilibrium the downward shift must be of equal size for both curves, as then (18) is equal to (19) and hence (20) must be equal to (21). Therefore stationary equilibrium labor input (and output) is not affected by a change in the rate of taxation of net income, as illustrated in Fig. 8 below.

Fig. 8

The fall in \( \tilde{p} \) is in this case purely a valuation phenomenon with no "real" implications. It reflects the decrease in value to the firm of a given stream of net income arising from an additional unit of labor, due to the fact that only 100(1 - \( \tau_1 \)) per cent of
this net income is now received by the owners. Simultaneously, however, the marginal opportunity cost of hiring falls by an equal amount, so no adjustment is needed on the part of the firm. The incidence of a tax on "Profit 1" is entirely on the owners of the firm, who suffer a decrease in its present value.

The instantaneous effect of an increase in the tax on Profit 1 — as can be seen from Fig. 8 — is a vertical jump from the old optimal path to a lower one. The interpretation of this jump is completely analogous to the stationary equilibrium one, and so we wind up with the rather surprising conclusion that the profits tax (or variations thereof) will affect neither the firm's optimal size in terms of inputs and output (even when all inputs are variable) nor its output decision and rate of growth on an optimal growth path. These conclusions are obviously at variance with established theory (see e.g. Musgrave [15], pp. 312-346 and Solow [23], pp. 337-339), which claims that while a profits tax does not affect the output decision when cost and revenue curves are given, it certainly does affect the long-run size of the firm and its optimal rate of growth. The reason for this divergence is our definition of "Profit 1". By this definition all costs were deductible from total revenue, irrespective of whether they were operating costs, \( w(x - y) \), or costs of acquiring additional resources, \( wy \), i.e. investment costs. Normally, however, outlays for investment are not immediately deductible for profit taxation purposes, but instead deductible in the form of depreciation allowances for a sequence of years after the investment is made.

In contrast to the first case we now assume that investment outlays are not deductible at all for purposes of profits taxation. We can call this new profit concept "Profit 2": total revenue minus operating costs. In this case the maximization problem of the firm becomes

\[
\max_{y} V_o = \int_{0}^{\infty} e^{-\rho t} \{[F(x - y) - w(x - y)](1 - \tau_2) - wy\} \, dt
\]
subject to the same constraints as before. This gives the new stationary conditions

\[ \frac{\delta \bar{y}}{\delta \tau_2} \bigg|_{x=0} = \left( [F'(x(1 - \gamma/\alpha)) - \nu](1 - \tau_2) + \nu \right)/\alpha \]

and

\[ \frac{\delta \bar{p}}{\delta \tau_2} \bigg|_{\bar{p}=0} = \nu/(\alpha - \rho - \gamma). \]

The effect of a change in the tax rate is

\[ \frac{\delta \bar{p}}{\delta \tau_2} \bigg|_{x=0} = \frac{\nu - F'(x(1 - \gamma/\alpha))}{\alpha} < 0 \quad \text{and} \quad \frac{\delta \bar{p}}{\delta \tau_2} \bigg|_{\bar{p}=0} = 0 \]

as illustrated in Fig. 9 below.

In this case, then, the profits tax does indeed have an effect on stationary equilibrium input and output of the firm. This is because the return on investment is due to taxation, while the cost of investment is not deductible, so changes in the tax rate will affect the rate of return on investment. This is the effect of a tax on profits which has been observed in the literature by inter alia Musgrave and Solow.
The instantaneous effects of this tax is a slowdown of the growth rate and an increase in output as team formation resources are shuffled over to the production department. The latter is, of course, once more a "perverse" effect of the type discussed earlier.

The "realistic" case is an intermediate one, but admittedly much closer to the first: investment expenditure is normally deductible in the form of depreciation allowances and therefore differs from the first case only by the interest losses due to postponement of deduction. But in the context of this model one can point to the arbitrariness of most actual tax systems with regard to deductibility rules. Investment in organization, marketing, training, research, etc. — sometimes called immaterial investment — is given a more favorable tax treatment than material investment, i.e. investment in buildings and physical equipment. To the extent that the two are complementary this may be of little importance. But it seems reasonable to assume that there is some substitutability between the two, and the consequent bias toward immaterial investment must be hard to defend on theoretical grounds.

Summarizing discussion

This ends our exercises in the comparative dynamics of the model. The main objective of the exposition has been to demonstrate that a model of the firm exhibiting all familiar "comparative static" properties may show a very different short-run behavior once its intertemporal trade-offs are taken into account. The main results are summarized in Table 2 below (where * refers to stationary equilibrium effects, and ∞ refers to instantaneous effects).

<table>
<thead>
<tr>
<th>Variable affected</th>
<th>Parameter increased</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>w α γ ρ τ₁ τ₂</td>
</tr>
</tbody>
</table>

- q* : - + ± - 0 -
- q∞ : + ± + 0 +
- x* : - ± ± 0 -
- x∞ : 0 0 0 0 0 0
- x∞ : - ± ± 0 -
While the effects on stationary equilibria $q^*$ and $x^*$ are all familiar from the literature, the effects on $\dot{q}$, $\dot{x}$ and $\dot{x}^*$ - the instantaneous effects - may not be so well known. It is self-evident, though, that *ex definitione* there can be no instantaneous effects on $x$ - the stock of effective labor. What is instantaneously affected is instead the rate of team formation, $\dot{x} + \gamma x$. These effects all have the "right" sign in terms of economic common sense. The "perverse" instantaneous effects on $q$ are better understood when seen as mirroring the effects on $\dot{x}$. The point is that team formation is a resource-consuming activity. With given resources ($x$ constant) $q$ must decrease when $\dot{x}$ increases and vice versa.

One key assumption behind these results (and in particular behind the "perverse" results) is the complete transferability of resources *inside* the firm. While the acquisition of new resources is a costly process, the use to which these resources are put can be changed without notice and at no cost. Labor - once turned into a team - can be freely moved between the team formation and production departments and is immediately fully productive in its new occupation. The actual possibility of "perverse" responses to parameter changes stands or falls with the realism of this assumption. In general this depends on whether the information and decision functions which must be imposed upon the individual members of the team are more firm-specific or more job-specific. If they are mainly firm-specific then it will be comparatively advantageous to shuffle team members around between different occupations inside the firm. If, on the other hand, these functions are mainly job-specific, then it would be more advantageous to hire new specialized labor, rather than retrain existing team members for a particular task. It is implied by the model presented here that the information and decision functions are entirely firm-specific with no job-specific elements for the two types of occupation.

These arguments, however, should not be permitted to disturb the essence of the comparative dynamics. The point is that a firm

11) The economic theory of teams defines these concepts more rigorously in terms of "specialization" (see Marschak-Radner [14], pp. 308-311).
which is induced by a parameter change to move toward a new stationary equilibrium will have to make internal adjustments to achieve that movement. These adjustments may produce a behavior very different from the one expected by comparison of the two equilibria.

VI. Concluding Remarks

The main purpose of the present paper has been to deduce the basic dynamics of the firm from maximization postulates. A dynamic problem appears because the firm can only increase production by investing in the coordination of new resources, thereby abstaining from present production. The analysis focuses on the simultaneity and interdependence of investment and production decisions. The focus on this interdependence has led to comparative dynamic results many of which may at first sight appear "perverse". These results are due to the oversimplifications which have been made in order to isolate dynamic trade-offs.

The model, of course, is no more a realistic description of the firm than a one-sector growth model is a realistic description of the economy. But just as a Ramsey model gives a basic understanding of the aggregate dynamics which must be faced by the politician, the model presented here is meant to be a first approximation of the corporate dynamics which must be met up to by the manager. Once this basic insight has been gained, the restrictive assumptions may be relaxed, and flesh can be put on the bones.

Among the many assumptions which are candidates for relaxation, only a few will be mentioned here. First, one may point out that the model as presented is completely deterministic with no elements of uncertainty. This is particularly embarrassing as it incorporates events which may lie in a very distant future. A testable model would certainly have to include uncertainty about future prices and technology.

Related to the certainty assumption is the assumption of a perfect capital market, which is implied in the maximization problem. The maximand may be negative for a long period of time when the firm - loosely
speaking - invests more than it produces. During this period wages must be paid out of funds borrowed at the constant rate of interest $\rho$. This assumption - that the rate of interest is independent of loan size and duration - may be reasonable under conditions of certainty, but not when future events (and hence the possibility of repayment) are uncertain. One obvious way of overcoming this restrictive assumption is to make the credit market monopsonistic so that the borrower bids up the interest rate against himself. Another way is to introduce a financial constraint e.g. of the form

$$F(x - y) - wx \leq kx$$

where $k$ is some rate-of-return requirement. $k \geq 0$ means that the firm is never allowed to borrow, so that it has to rely completely on internal finance. For $k > w$ the firm will normally have to follow a financially constrained path in the early stage of growth. During that stage the firm will grow at a maximum rate subject to the rate-of-return requirement.\(^{12}\)

The assumptions that the firm sells one single homogeneous output and buys one single homogeneous input on perfectly competitive markets may seem unduly restrictive for all practical purposes. The formulation (4), however, lends itself to a wide range of interpretation. For example, assuming that information and decision functions are also job-specific, $X$ can be taken to be a vector $(x_1, x_2, \ldots, x_n)'$ of specialized resources, the maximum income of which in team production is $F(X)$.\(^{13}\) Particular elements of $X$ may be stocks of effective specialized labor (like engineers, carpenters, etc.) but also operating specialized machines and equipment, and perhaps even "intangibles" like technology, goodwill, etc. The growth of this

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\(^{12}\) It can be noted that during that stage the behavior of growth maximizing and profit (present value) maximizing firms - a distinction which is sometimes made in the theoretical literature (see e.g. Solow [23]) - will be identical.

\(^{13}\) In a recent paper Rubin [21] has presented such a multifactor (and multiproduct) model of the firm, based very much on the same ideas as the present model. He also arrives to similar conclusions in some respects. Staying within a period-analytic framework, however, he is not in a position to perform an explicit treatment of the dynamic trade-offs.
system could be described by

\[ \dot{X} = AY - \Gamma X \]

where \( Y \) is a vector \((y_1, y_2, \ldots, y_n)'\) of resources devoted to hiring. \( A \) is a "growth technology" matrix \([a_{ij}]\), and \( \Gamma \) is a vector of depreciation rates \((\gamma_1, \gamma_2, \ldots, \gamma_n)'\). The control is the vector \( Y = (X + \Gamma X)A^{-1} \), and it should be chosen so that the marginal increase of the present value of the firm is equal to the marginal opportunity cost \( F'(X - Y) \) of maintaining this rate of growth. Disregarding all technical problems involved, such a "general equilibrium" model of the firm could in principle be made to handle simultaneously, e.g. investment in goodwill by means of advertising (cf. Gould [8]), investment in cost reductions (technological knowledge) by means of research (cf. Nordhaus [16]), investment in temporary monopolies by means of product development (cf. Kamien and Schwartz [10]), even "investment in market share" by means of price reductions (cf. Phelps and Winter [16]), and "investment in a lower quit rate" by means of higher wages (cf. Salop [22]).

Though not meant for direct application the model as it stands does have some suggestive implications, of which a few can be mentioned here. First, the analysis helps to explain why expansion by merger may be more profitable than internal expansion (recruitment and training of new labor). As the organization of resources into a productive team (like the firm) is a costly investment, the owners of a growing firm in need of more than marginal additions to its present stock of resources will be willing to pay a premium on a team relative to resources, which have first to be turned into a team. This premium is given by \( \bar{p} \). Normally, however, the information and decision structures of one firm are not directly effective in another firm, so that some reorganization costs must be deducted from the premium. The important point, though, is that the two firms need not have the same information structure; it is enough that the central decision-making units have an information system in common. Thereby the team formation investment which is sunk in a firm will make it command a premium in value over the sum of its factors of production evaluated at market prices.
Second, the analysis suggests that small firms may have difficulties in competing with large firms when new productive opportunities open up. If it is subject to a financial constraint it may not seem worthwhile to a small firm to enter into competition with larger firms, because of the high costs of growth. In particular, this point emphasizes the role of well-functioning capital markets for the "equality of opportunity" in the competitive process.

Third, there are implications for the measurement of corporate productivity as discussed on p. 17 above. Rapidly growing firms and industries will have a lower observed factor productivity ($q/x$) than slowly growing or stationary ones. This, of course, is no sign of inefficiency, but may to the contrary be a sign of dynamic efficiency. To the extent that factors of production are specialized in either production or hiring and can be easily identified, this does not cause too much trouble. The appropriate measure of productivity in direct production will then be $q/(x-y)$. If, on the other hand, the hiring resources are not easily identified, or else if we want to measure the overall productivity of factors of production, then the best measure of the firm's technical efficiency may be $(\dot{x}+q)/x$, where, however, the aggregation of the rate of growth and the flow of output causes obvious weighting problems.

Finally, the analysis has some bearing on economic policies which are meant to stimulate economic activity. Consider a parameter change which does in fact increase the stationary equilibrium of the firm (e.g. a lowering of the rate of interest or the real wage), and which therefore is inferred from comparative statics to have a positive effect on output. While such a parameter change does indeed increase the rate of growth of the firm, it may — as we have seen — affect output negatively in the short run. With several specialized factors of production the latter effect is less likely, because in this case the resources employed in the production department cannot always be used in the team formation department. It is clear, however, that the main effect of "stimulating" policies on already growing firms is a shift over to more long-run activities. Hence, a firm which is contemplating investment in physical equipment might respond to a cut in the rate of interest by reducing capital expenditure and spending more on research and development.
At any rate, an expansion of output will not always or even normally be the immediate effect of a "stimulating" parameter change on a growing firm. In analogue with the growing economy the growing firm will have its von Neumann path along which it grows at a maximum rate, spending all its resources on investment in recruitment, training, organization, product and market research, etc. In early stages of growth the firm will be on a path very close to the von Neumann ray and then gradually move away from it, devoting a continuously larger share of its resources to current production. The effect of a parameter change will - as we have seen in Section V above - depend on whether it shifts the optimal path to a position closer to or further away from the maximum growth path.

The simplicity of the model presented in this paper has already been emphasized. It is not meant for straightforward application or direct empirical testing. But the analysis seems to give a reasonable starting point for our fundamental understanding of the workings of a production sector of the economy with the following characteristics:

- The production system consists of several independent firms.
- The organization of production within a firm is a resource-consuming activity.
- Firms have not by far reached their stationary equilibria.

To the extent that these three points catch important properties of real world production systems, it seems that the type of dynamic microtheory presented here could provide a useful basis for the construction of aggregative models of production.

\[14\] In our simple one-factor model this path is represented by \( y=x \) (see Fig. 3).
REFERENCES


APPENDIX I

Derivation of necessary and sufficient optimality conditions by the Maximum Principle

The following two theorems are taken directly from Arrow-Kurz\(^a\) (pp. 48-49), but the notation has been changed to the one used in the paper.

**Theorem I** (Necessary conditions)

Let \( y^*(t) \) be a choice of instrument \((t \geq 0)\) which maximizes

\[
\int_0^\infty e^{-\delta t} [F(x(t) - y(t)) - wx(t)] \, dt
\]

subject to the condition

(a) \( \dot{x} = \alpha y(t) - \gamma x(t) \),

a constraint

(b) \( x(t) - y(t) \geq 0 \),

an initial condition on the state variable and the non-negativity condition

(c) \( x(t) \geq 0 \)

on the state variable. Then there exists a function of time, \( \tilde{p}(t) \), such that for each \( t \)

(d) \( y^*(t) \) maximizes \( H(x(t), y, \tilde{p}(t), t) \) subject to the constraint (b) and the additional constraint \( \alpha y - \gamma x(t) \geq 0 \) for \( x(t) = 0 \), where \( H(x, y, \tilde{p}, t) = F(x - y) - wx + \tilde{p}(\alpha y - \gamma x) \)

(e) \( \dot{\tilde{p}} = \rho \tilde{p} - \delta L/\delta x \), evaluated at \( x = x(t), y = y^*(t) \),

\( \tilde{p} = \tilde{p}(t) \), and

(f) \( L(x, y, \ddot{p}, \ddot{q}, \ddot{r}, t) = H(x, y, \ddot{p}, t) + \ddot{q}(x - y) + \ddot{r}(\alpha y - \gamma x) \) and the Lagrange multipliers are such that

(g) \( \frac{\partial L}{\partial y} = 0 \) for \( x = x(t), y = y^*(t), \ddot{p} = \ddot{p}(t) \)
\( \ddot{q}(t) \geq 0, \ddot{q}(t)[x(t) - y^*(t)] = 0 \)
\( \ddot{r}(t) \geq 0, \ddot{r}(t)x(t) = 0, \ddot{r}(t)[\alpha y^*(t) - \gamma x(t)] = 0 \)

**Theorem II** (Sufficient conditions)

If \( J^0(x, \ddot{p}, t) = \max_y H(x, y, \ddot{p}, t) \)

is a concave function of \( x \) for given \( \ddot{p} \) and \( t \), then any policy is optimal that satisfies the conditions of Theorem I and the transversality conditions

\[
\lim_{t \to \infty} e^{-\rho t} \ddot{p}(t) \geq 0 \quad \lim_{t \to \infty} e^{-\rho t} \ddot{p}(t)x(t) = 0
\]

Necessary conditions are derived from Theorem I (e) and (g). Write

\[ L = F(x - y) - wx + \ddot{p}(\alpha y - \gamma x) + \ddot{q}(x - y) + \ddot{r}(\alpha y - \gamma x) \]

Then

\[
\frac{\delta L}{\delta x} = F'(x - y) - w - \ddot{p}\gamma + \ddot{q} - \ddot{r}\gamma = \rho \ddot{p} - \ddot{r}
\]

and

\[
\frac{\delta L}{\delta y} = -F'(x - y) + \ddot{p}\alpha - \ddot{q} + \ddot{r}\alpha = 0
\]

Rearranging we get

\[
\ddot{p} = w - F'(x - y) + (\rho + \gamma)\ddot{p} - \ddot{q} + \ddot{r} \quad \text{(AI:1)}
\]

and

\[
\ddot{p} = (F'(x - y) + \ddot{q})/\alpha - \ddot{r} \quad \text{(AI:2)}
\]

Furthermore, from Theorem I (g)

\[ \ddot{q} \geq 0, \quad \ddot{q}(x - y) = 0 \quad \text{(AI:3)} \]

\[ \ddot{r} \geq 0, \quad \ddot{r}x = 0, \quad \ddot{r}(\alpha y - \gamma x) = 0 \quad \text{(AI:4)} \]
As we are only considering situations where \( x > 0 \), \( \bar{z} = 0 \) from (AI:4). Thereby equations (AI:1), (AI:2), and (AI:3) reduce to (7), (8), and (9) in the main text.

Sufficiency is established as for any given \( \bar{p} \) and \( y \) the maximand (by assumption) is concave and the constraints linear in \( x \).
APPENDIX II

The shape of the $y = 0$ schedule in the $(x, y)$ phase

From the main text we have

$$
\dot{y} = \dot{x} - \frac{F'_{\alpha}(x-y)}{F''(x-y)}
$$

(16)

Furthermore, from (7:a) and (8:a)

$$
\dot{p} = w - F'(x-y)[1 - (\rho + \gamma)/\alpha]
$$

(AII:1)

and

$$
\dot{x} = \alpha y - \gamma x
$$

(2)

Substituting (2) and (AII:1) into (16) we get

$$
\dot{y} = \alpha y - \gamma x - \frac{\{w - F'(x-y)[1 - (\rho + \gamma)/\alpha]\} \alpha}{F''(x-y)} = \Phi(x, y)
$$

(AII:2)

Now, we know that

$$
\frac{d\Phi}{dx} \Big|_{y=0} = -\frac{\delta \Phi}{\delta x}
$$

We therefore take first

$$
\frac{\delta \Phi}{\delta y} = (\rho + \gamma) - G(x - y)
$$

where

$$
G(x - y) = \frac{\{w - F'(x-y)[1 - (\rho + \gamma)/\alpha]\} \alpha}{F''(x-y)} F''(x-y)
$$

\[\frac{[F''(x-y)]^2}{[F''(x-y)]^2}\]
The sign of \(G(x - y)\) is opposite to the sign of \(F''(x - y)\), as the expression inside the big brackets is negative by assumption (we are only studying a growth path) and \(\alpha\) and the squared denominator are positive. It is consistent with the Inada conditions to assume that \(F''(x - y) > 0\), and we shall retain this assumption. As a result \(G(x - y) < 0\) and hence \(\frac{\delta \phi}{\delta y}\) always.

Next,

\[
\frac{\delta \phi}{\delta x} = \alpha - (\rho + 2\gamma) + G(x - y)
\]

We have assumed (p. 9 in the main text) that \(\alpha - \rho + 2\gamma > 0\), and we have just seen that \(G(x - y) < 0\), so the sign of \(\frac{\delta \phi}{\delta x}\) is ambiguous, and depends on the relative size of these two expressions. We can make the simplifying further assumption that

\[
\frac{F''(x - y)}{[F''(x - y)]^2} = \text{constant \ all \ (x - y)}
\]

In this particular case we get the relation depicted in Fig. A 1, where \(\frac{\delta \phi}{\delta x}\) shifts sign from negative to positive as \(x\) increases.

Fig. A 1
for any given \( y = \bar{y} \). Let the switching point be \( \bar{x} \). Then

\[
\begin{align*}
\frac{\delta \phi}{\delta x} &= \begin{cases} 
< 0 & \text{for } x < \bar{x} \\
0 & \text{for } x = \bar{x} \\
> 0 & \text{for } x > \bar{x}
\end{cases} 
\end{align*}
\]

and therefore

\[
\left. \frac{dy}{dx} \right|_{y=0} = -\frac{\delta \phi}{\delta x} = -\frac{a - (\rho + 2\gamma) + G(x - y)}{(\rho + \gamma) - G(x - y)} \begin{cases} 
> 0 & \text{for } x < \bar{x} \\
= 0 & \text{for } x = \bar{x} \\
< 0 & \text{for } x > \bar{x}
\end{cases}
\]

as shown in Fig. 3 in the main text. Furthermore we see that

\[
\lim_{(x-y) \to (x^* - y^* )} G(x - y) = 0
\]

so that

\[
\left. \frac{dy}{dx} \right|_{y=y^*} = -\frac{a - (\rho + 2\gamma)}{\rho + \gamma} < 0
\]

and therefore \( x < x^* \) as shown in Fig. 3.

A special case arises for a quadratic production function, where

\[
G(x - y) = F''(x - y) = 0.
\]

In this case

\[
\left. \frac{dy}{dx} \right|_{x=0} = -\frac{a - (\rho + 2\gamma)}{\rho + \gamma} < 0
\]

so the \( y=0 \)-schedule becomes a negatively sloped straight line in the \( (x, y) \) phase, as illustrated in Fig. A2 below.
Under these circumstances the optimal solution will be boundary with \( y=x \) up to a "switching point" \( y'>y^* \). On this boundary path the Kuhn-Tucker multiplier \( \bar{q} \) (see Appendix I) will take a non-zero value. When \( y' \) is reached then \( \bar{q}=0 \), and \( y \) starts to fall on an unconstrained growth path.