Closed Timelike Curves in Exact Solutions

Bachelor Degree Project

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Abstract
This project aims to study general relativity to the extent to understand the occurrence and behaviors of closed timelike curves (CTCs) in several exact solutions of Einstein’s field equations. The rotating black hole solution, the Gödel universe and the cosmic string solutions are studied in detail to show how CTCs arise in these spacetimes. The chronology-violating paradoxes and other unphysical aspects of CTCs are discussed. The spacetimes where CTCs arise possess properties which are argumented to be unphysical, such as lack of asymptotic flatness and being infinite models. With quantum computational networks it is possible to resolve the paradoxes which CTCs evoke. With all these attempts of resolving CTCs, our conclusion is that CTCs exist quantum mechanically, but there is a mechanism which inhibits them to be detected classically.

Sammanfattning
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1 Introduction

In the period before Aristotle, the Earth was generally believed to be a flat disk. Sailing far enough on the sea would eventually lead to an edge and a plummet into vast nothing. It turned out however that no matter how far the ship went, the edge never came and eventually you would get back to the point of initial location. However, no matter how emancipated today’s society is, looking out far into the beautiful landscape from a tall mountain would still make you believe that Earth is flat. The illusion arises from local flatness. The Earth is indeed flat even through the glasses of modern physics: but only locally, close enough to the point of examination.

The theory of special relativity encourages us to regard time as a component of our world: it is not enough to merely localize an event, we also need to specify the time of it. Time together with spatial components gives us the four dimensional spacetime of our observable world. Special relativity explores the nature of flat spacetime: a spacetime which is not curved. Now flatness in the sense of relativity no longer pinpoints the same fact as Aristotle’s notion of flatness: but rather, flatness in relativity describes spacetimes possessing triangles with angles summing up to \( \pi \), Pythagoras theorem, and all other Euclidean notions of space, as we are accustomed to, as well as the conventional relation between time, space and velocity. It is nearly just a mere coincidence that the two labels should coincide.

Up until the time of Albert Einstein, like in the analogy of Aristotle’s discovery, spacetime was thought to be flat. In his published works during 1915, Einstein proposed other spacetimes: he suggested there should be curvature due to gravity. In this theory of general relativity, it turns out that matter indeed gives rise to changes in spacetimes: it leads to curvatures. Following the development of general relativity, a series of different solutions of spacetimes emerged, each being a model of matter distributions.

In the theory of general relativity, one can dig in and solve the equations exactly, analytically. But these solutions reflect barely toy models of our real world. In theory, one could describe the entire universe with the equations proposed, if only one knew all matter which it consists of, and if one has almost infinite capacities on numerical calculators. In many cases, however, it is sufficient to cast an eye on the toy models. Black holes are one example of these toy models which came about as a very simple solution to the equations and were at their day of birth considered to be peculiarities of general relativity and mathematics. Today, we know of numerous black holes and ways of measuring them: from paper and pen they became reality and observations.

Similarly, another wit of mathematics are the closed timelike curves, CTCs. These come about in numerous solutions to the equations of general relativity. CTCs provide in theory a way of time traveling - backwards. As special relativity indeed permits time travel forward in time, it does, as we know it, forbid any backward time travel, which would and does lead to theoretical paradoxes of causality: the notion of causes and consequences. CTCs are therefore vulnerable objects by the physicists’ society: would they become reality as black holes did, then much of today’s view on causality would crumble and urge for new ways out.

Some of the most famous solutions of the equations where CTCs arise are the Gödel universe, Gott’s cosmic strings and rotating black holes. The aim of this thesis is to examine how CTCs arise in these exact solutions of Einstein’s field equations, and study the paradoxes which they evoke classically. Quantum mechanics is believed to resolve these paradoxes, which the second part of this project aims to study. The project is a literature study, basing the information on books and lecture notes in general relativity and numerous articles on the topic of CTCs.
2 General relativity

In the first part of the report, we encounter those elements of general relativity which are inevitable for the understanding of closed timelike curves (CTCs). We will equip ourselves with some mathematical tools, such as manifolds, tensors, and derivatives and then consider Einstein’s field equations (EFEs), which are at the core of general relativity.

2.1 Spacetimes as mathematical manifolds

Spacetime is the merging of spatial and temporal coordinates. A trajectory in the spacetime is referred to as a worldline (or also just trajectory, path or ray). Firstly, we need to see what spacetime actually is mathematically and how we equip it with properties which will satisfy our conditions for spacetime as we can measure it in reality. The introduction to general relativity in this part is based on [1] and [2].

2.1.1 Manifolds

To be able to analyze different spacetimes, we need to impose some conditions for them. Firstly, we need them to be locally flat, locally Minkowski. This means that there is a coordinate set locally around every point, called a locally inertial frame, in which the metric of the spacetime is the Minkowski metric, $\eta_{\mu\nu}$. We also claim for these local flatnesses to be patched together smoothly. These criteria considered mathematically dwell in differentiable manifolds.

A differentiable manifold is necessarily a topological manifold. A topological manifold is characterized by a set $T$ and a set $C$ of open subsets $\{S_n\}$ of $T$, $S_n \subseteq T \ \forall n$. This set $C$ must satisfy the following:

1. $\varnothing, T \in C$
2. for any (finite or infinite) collection of the subsets:
   $$ \bigcup_n S_n = T $$
3. for any finite collection of the subsets:
   $$ \bigcap_{i=1}^{k} S_i \in C $$
4. for every subset $S_n$, there is an injective continuous map $\phi_n$ with a continuous inverse map $\phi_n^{-1}$:
   $$ \phi_n : S_n \Rightarrow \mathbb{R}^d $$
   where $d$ is then called the dimension of the manifold.
5. for any two overlapping subsets $S_i$ and $S_j$, we demand the transition function defined by
   $$ \phi_i \circ \phi_j^{-1} : \phi_j(S_i \cup S_j) \Rightarrow \phi_i(S_i \cup S_j) $$
to be continuous.

For a differentiable manifold we further impose that the transition functions be smooth,

1. \( \phi_i \circ \phi_j^{-1} \in C^\infty \quad \forall i, j = 1, \ldots, n \) with \( S_i \cap S_j \neq \emptyset \)

### 2.1.2 Tensors

The **tangent space** \( T_p M \) of a manifold \( M \) at a point \( p \in M \) is defined to be the set of all tangent vectors of all curves on the manifold through the point. The tangent space is a vector space, meaning that it has a basis in which we can express any element as the linear combination of the basis, with the coefficients of the expansion called the components.

On a (differentiable) manifold, a natural choice for the basis of the tangent space in some coordinate system is the partial derivatives with respect to the coordinates. If we let the index \( \mu \) denote the coordinates, then our basis is

\[
\frac{\partial}{\partial x^\mu} := \partial_\mu, \quad (2.1)
\]

where \( \mu = 1, \ldots, d \), or often \( \mu = 0, \ldots, d - 1 \).

An arbitrary vector \( V \in T_p M \) can thus be expressed as

\[
V = V^\mu \partial_\mu. \quad (2.2)
\]

Note that repeated indices located at a lower and a higher position are always summed over. This is Einstein’s summation convention and is used throughout the report.

We define the dual tangent space \( T_p^* M \) to be the set of all linear maps \( \omega \) of vectors in \( T_p M \) to the real numbers. These \( \omega \) are called the dual forms, \( \omega \in T_p^* M \),

\[
\omega : T_p M \to \mathbb{R}. \quad (2.3)
\]

The basis set \( b^\mu \) of the dual space is such that:

\[
b^\mu (\partial_\nu) = \delta^\mu_\nu. \quad (2.4)
\]

Suggestively, this will thus be \( b^\mu = dx^\mu \), so an arbitrary dual form can be written as

\[
w = w_\nu dx^\nu. \quad (2.5)
\]

Dual forms and vectors are a simple type of a general tensor. A tensor \( T \) of type \( (m, n) \) is a linear map from \( m \) dual tangent spaces and \( n \) tangent spaces to the real numbers:

\[
T : T_p^* M \times \ldots \times T_p^* M \times T_p M \times \ldots \times T_p M \to \mathbb{R}, \quad (2.6)
\]

and which can be written as linear combinations of the bases of the two kinds of spaces, with according number of bases

\[
T = T^{\alpha_1, \ldots, \alpha_m}_{\beta_1, \ldots, \beta_m} dx^{\beta_1} \ldots dx^{\beta_m} \partial_{\alpha_1} \ldots \partial_{\alpha_m}. \quad (2.7)
\]
The components of tensors should transform under any coordinate change

\[ x^\mu \rightarrow x'^\mu, \]  

as the following tensor transformation law:

\[ T^{\alpha_1' \ldots \alpha_n'}_{\beta_1' \ldots \beta_m'} = \frac{\partial x^{\alpha_1}}{\partial x'^{\alpha_1}} \ldots \frac{\partial x^{\alpha_n}}{\partial x'^{\alpha_n}} \frac{\partial x^{\beta_1}}{\partial x'^{\beta_1}} \ldots \frac{\partial x^{\beta_m}}{\partial x'^{\beta_m}} T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m}. \]  

(2.8)

It is fully legitimate to measure the distance on a curved surface, such as a cylindrical box, if we take our sewing tape and wind it around the curved shaped object, then read off the scales given on the tape. In a general curved spacetime, we can measure distances mathematically with the help of the metric. We define the infinitesimal line element, or simply the line element (squared) as

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \]  

(2.9)

We define the proper time \( \tau \) to be the time measured by a frame of a particle traveling on some worldline. The particle will be at rest in its own frame and so the spatial basis one-forms will all vanish in the line element, leaving us with only the time differential. Due to a sign convention of the line element and metric, we define the proper time to be minus the line element,

\[ d\tau^2 = -ds^2 = -g_{\mu\nu} dx^\mu dx^\nu. \]  

(2.10)

The norm of a vector is defined to be the inner product with itself. The inner product of two vectors is defined as the action of the metric on the two vectors

\[ g : T_p M \times T_p M \rightarrow \mathbb{R}, \]

\[ g(V, W) = g_{\mu\nu} V^\mu W^\nu = V_\mu W^\mu. \]  

(2.11)

The norm of a vector is then

\[ g_{\mu\nu} V^\mu V^\nu = V^\mu V_\mu. \]  

(2.12)

2.1.3 Covariant derivatives

We note that the usual partial derivative of the components of a vector does not transform as a \((0, 1)\) type tensor. This we see by taking the partial derivative of the vector component and plugging in the transformation law for the vector component,

\[ \partial_\rho V^\nu = \partial_\rho \left( \frac{\partial x^\mu}{\partial x'^\rho} V'^\nu \right) = V'^\nu \frac{\partial}{\partial x'^\rho} \left( \frac{\partial x^\mu}{\partial x'^\rho} \right) + \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial}{\partial x'^\rho} \left( V'^\nu \right) = \]

\[ = V'^\nu \frac{\partial}{\partial x'^\rho} \left( \frac{\partial x^\mu}{\partial x'^\rho} \right) + \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial}{\partial x'^\rho} \left( V'^\nu \right) = \]

\[ = V'^\nu \frac{\partial}{\partial x'^\rho} \left( \frac{\partial x^\mu}{\partial x'^\rho} \right) + \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial}{\partial x'^\rho} \left( V'^\nu \right). \]  

(2.13)
The underlined expression corresponds to the usual transformation law for tensors, if we would regard \( \partial_\mu V^\nu \) as a \((1,1)\)-type tensor. However we see that it has an additional term, which leads to it not being a tensor. Therefore we introduce a **covariant derivative** \( \nabla_\rho \) which takes a vector and transforms it into a \((1,1)\)-type tensor. This is done by adding to the partial derivative a correction term \( \Gamma^\mu_{\rho\lambda} \), which we call the **connection coefficients**, according to

\[
\nabla_\rho V^\mu = \partial_\rho V^\mu + \Gamma^\mu_{\rho\lambda} V^\lambda, \quad (2.17)
\]

where we add a linear term of a \( d \times d \) matrix for each component of the vector, where \( d \) is the dimension of our manifold. These connection coefficients cancel the non-tensorial transformation of the partial derivative, so that the covariant derivative will transform as a tensor. To check this, we impose that the covariant derive of a vector should transform as a \((1,1)\)-type tensor and see what the connection coefficients have to be. If \( \nabla_\rho V^\mu \) is considered a tensor, then it should transform as

\[
\nabla_\rho' V'^\mu = \partial_\rho' V'^\mu + \Gamma'^\mu_{\rho'\lambda'} V'^\lambda' = \nabla_\rho V^\mu = \partial_\rho V^\mu + \Gamma^\mu_{\rho\lambda} V^\lambda, \quad (2.18)
\]

but according to the definition of the covariant derivative (2.17), we can write this alternatively as

\[
\nabla_\rho' V'^\mu = \partial_\rho V^\mu + \Gamma'^\mu_{\rho'\lambda'} V'^\lambda' = \nabla_\rho V^\mu = \partial_\rho V^\mu + \Gamma^\mu_{\rho\lambda} V^\lambda, \quad (2.19)
\]

Setting expressions (2.18) and (2.19) equal to each other gives

\[
\frac{\partial x^{\mu'}}{\partial x^p} \frac{\partial x^p}{\partial x^{\rho'}} + \frac{\partial x^{\rho'}}{\partial x^p} \Gamma^\mu_{\rho\lambda} V^\lambda = \frac{\partial x^{\mu'}}{\partial x^p} \frac{\partial x^p}{\partial x^{\rho'}} V^\mu + \frac{\partial x^{\rho'}}{\partial x^p} \left( \frac{\partial x^{\mu'}}{\partial x^p} \right) + \Gamma'^\mu_{\rho'\lambda'} V'^\lambda'. \quad (2.20)
\]

Plugging in the final transformation law for the only primed vector yields

\[
\frac{\partial x^{\mu'}}{\partial x^p} \frac{\partial x^p}{\partial x^{\rho'}} \Gamma^\mu_{\rho\lambda} V^\lambda = \frac{\partial x^{\mu'}}{\partial x^p} V^\mu \frac{\partial \partial x^{\rho'}}{\partial x^p} \left( \frac{\partial x^{\mu'}}{\partial x^p} \right) + \Gamma^\mu_{\rho\lambda} \frac{\partial x^{\lambda'}}{\partial x^p} V^\lambda. \quad (2.21)
\]

In the first term of the right hand side of the equation we rename the dummy index \( \mu \rightarrow \lambda \) and claim that this is to hold for all vectors \( V^\lambda \), and so the coefficients on either side for the vectors should be equal,

\[
\frac{\partial x^{\mu'}}{\partial x^p} \frac{\partial x^p}{\partial x^{\rho'}} \Gamma^\mu_{\rho\lambda} = \frac{\partial x^{\mu'}}{\partial x^p} \frac{\partial \partial x^{\rho'}}{\partial x^p} \left( \frac{\partial x^{\mu'}}{\partial x^p} \right) + \Gamma^\mu_{\rho'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^p} V^\lambda. \quad (2.22)
\]
Let us now multiply this equation with a partial derivative \( \frac{\partial x^n}{\partial x^\beta} \),
\[
\Gamma^\mu_\rho^\lambda \frac{\partial x^\alpha}{\partial x^\beta} \frac{\partial x^\lambda}{\partial x^\beta} = \Gamma^\rho_\mu^\lambda \frac{\partial x^\rho}{\partial x^\beta} \nabla^\rho \Gamma^\mu_\rho^\lambda.
\]
\[
\Rightarrow \Gamma^\mu_\rho^\lambda, \delta^\rho_\beta \delta^\alpha_\lambda = \frac{\partial x^\alpha}{\partial x^\beta} \frac{\partial x^\lambda}{\partial x^{\rho'}} \Gamma^\mu_\rho^\lambda - \frac{\partial x^\alpha}{\partial x^{\rho'}} \frac{\partial x^\rho}{\partial x^\beta} \partial \frac{\partial x^{\rho'}}{\partial x^\lambda}.
\]
\[
\Rightarrow \Gamma^\mu_\rho^\lambda = \frac{\partial x^\lambda}{\partial x^{\rho'}} \frac{\partial x^{\rho'}}{\partial x^{\rho'}} \Gamma^\mu_\rho^\lambda - \frac{\partial x^\alpha}{\partial x^{\rho'}} \frac{\partial x^\rho}{\partial x^\beta} \partial \frac{\partial x^{\rho'}}{\partial x^\lambda}.
\]
This is finally the transformation law for the connection coefficients. We see that they do not transform as a tensor, given by (2.9), and so the coefficients are not components of a tensor (a reason why they are not called the connection tensor).

However, the difference between two connection coefficients, with the lower indices interchanged, is a tensor, as the nontensorial part of the transformation law cancels (this nontensorial part is underlined in the transformation law and is symmetric in the interchange of \( \rho' \leftrightarrow \lambda' \)). This difference is called the torsion, defined as
\[
T^\mu_\rho^\lambda = \Gamma^\mu_\rho^\lambda - \Gamma^\mu_\rho^\lambda.
\]
Most manifolds representing spacetimes are torsion-free, which means that the torsion components all vanish and so the connection coefficients are symmetric in their lower indices (if we extend the notion of symmetry to non-tensors as well).

Covariant derivatives can also act on one-forms and an arbitrary \((k, l)\)-type tensor. From the additional criterion that the covariant derivative reduces to the partial derivative on scalars (functions), we get that the covariant derivative of a one-form is given by
\[
\nabla_\rho \omega_\mu = \partial_\rho \omega_\mu - \Gamma^\lambda_\rho^\mu \omega_\lambda,
\]
and generally, we can map a \((m, n)\)-type tensor to a \((m, n + 1)\)-type tensor as
\[
\nabla_\rho T^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n} = \partial_\rho T^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n} + \Gamma^{\mu_1}_{\rho \sigma} T^{\sigma \ldots \mu_m}_{\nu_1 \ldots \nu_n} + \ldots + \Gamma^{\mu_m}_{\rho \sigma} T^{\mu_1 \ldots \sigma}_{\nu_1 \ldots \nu_n} - \Gamma^{\sigma}_{\rho \sigma} T^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n} - \ldots - \Gamma^{\sigma}_{\rho \sigma} T^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n}.
\]
Now there are several connections possible, since we can choose different connection coefficients. However, there is one choice which assures that it is torsion-free, so that the connection coefficients are symmetric, and that the covariant derivative is metric compatible, which means that for all indices we have
\[
\nabla_\rho g_{\mu \nu} = 0.
\]
This choice of the connection coefficients is called the Levi-Civita connection or the Christoffel symbols. The expression for it can be obtained simply by permuting the indices of the metric-compatibility condition (2.27) in all even permutations, expressing the covariant derivatives, and taking a linear combination of them and using the symmetry in the lower indices of the connection coefficients, giving
\[
\Gamma^\mu_{\rho \sigma} = \frac{1}{2} g^{\mu \lambda} (\partial_\rho g_{\lambda \nu} + \partial_\sigma g_{\lambda \nu} - \partial_\lambda g_{\rho \nu}).
\]
2.1.4 Geodesics and the Riemann tensor

Given a manifold with a set of tangent spaces and tangent vectors in the tangent spaces, how can we move a vector from one point in the manifold to another point? Is this done in a unique way, independent of the path of movement we choose? The answer is no: the notion of parallel transport in differential geometry describes how a vector can be moved, but that this is actually a path-dependent notion.

Parallel transporting a vector \( V \) defined on a manifold with dimension \( d \), along a curve \( x^\mu(\lambda) \) means that the vector components do not change along the curve as \( \lambda \) evolves,

\[
\frac{d}{d\lambda} V^\mu = 0 \quad \forall \mu = 1, ..., d,
\]

which can be written with the chain rule as

\[
\frac{dx^\nu}{d\lambda} \frac{\partial}{\partial x^\nu} V^\mu = 0,
\]

and to have a tensorial equation, we replace the partial derivative with the covariant derivative, upon which we receive the equation of parallel transport,

\[
\frac{dx^\nu}{d\lambda} \nabla_\nu V^\mu = 0.
\]

A geodesic of a spacetime is such a curve which parallel transports its own vector, so the vector \( V \) in the above equation of parallel transport (2.31) can be replaced with the tangent vector of the curve \( x^\mu \) and expanding the covariant derivative by its definition gives us the geodesic equation,

\[
\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0.
\]

Let us now look at the deviation of parallel transporting a vector along two different paths. Consider an infinitesimal parallelogram ABCD on a manifold. We can approximate the sides of the parallelogram to be straight, if we choose them small enough. A vector \( V \) at point A is given and we wish to transport it to the point D. We can either go the way through B or the way through C. Let the vector we receive by going through C be called \( V^C \), and similarly the vector through B be called \( V^B \), see Fig. 1. Since now the two vectors are elements of the same tangent space, it makes sense to take the difference of these. The difference, \( \delta V \), will be proportional to the area of the parallelogram, the vector and to the Riemann tensor \( R^\mu_{\rho\sigma\nu} \). In components, we can write this as

\[
\delta V^\mu = R^\mu_{\rho\sigma\nu} A^\rho \delta (AB)^\sigma \delta (AC)^\nu,
\]

where \( \delta (AB) \) and \( \delta (AC) \) denote the infinitesimal lengths of the sides of the parallelogram between the denoted points. The Riemann tensor is a \((1, 3)\)-type tensor, which maps three vector fields to a fourth vector field. By explicitly calculating the vector \( V \) going through the two paths, one finds the expression for the Riemann tensor to be

\[
R^\mu_{\rho\sigma\nu} = \partial_\sigma \Gamma^\mu_{\rho\nu} - \partial_\nu \Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\sigma\lambda} \Gamma^\lambda_{\rho\nu} - \Gamma^\mu_{\nu\lambda} \Gamma^\lambda_{\rho\sigma}.
\]

Out of a \((1, 3)\)-type tensor as the Riemann tensor, we can contract the upper and one lower index to obtain a \((0, 2)\)-type tensor. We define the Ricci tensor \( R_{\mu\nu} \) as the contracted Riemann tensor according to

\[
R_{\mu\nu} = R^{\sigma\mu}_{\rho\sigma\nu} = g^{\sigma\lambda} R_{\lambda\mu\sigma\nu}.
\]
Figure 1: An infinitesimal parallelogram ABCD on some manifold. Sides of parallelogram can be approximated to be straight lines where usual Euclidean geometry can be used. A vector \( V \) defined at point A can be moved to point D in two manners: via the red way through B, or via the blue way through C. The difference of the resulting vectors \( V_B \) and \( V_C \) at point D is proportional to the Riemann tensor.

Further we can contract the Ricci tensor to build the **Ricci scalar** \( R \),

\[
R = R^\lambda{}_{\lambda} = g^{\lambda\mu} R_{\mu\lambda}. \tag{2.36}
\]

The Riemann tensor defines the curvature of a manifold: it describes the deviation of the transported vectors along two different paths. In flat spacetime, a vector can be transported in any path yielding the same result, in which case the Riemann tensor vanishes, giving vanishing Ricci tensor and Ricci scalar.

For convenience one defines the **Einstein tensor** \( G_{\mu\nu} \) as

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{2.37}
\]

### 2.1.5 Curves and causality

Curves will play a major role in this thesis, and it is worth defining them more rigorously. A curve \( \mathcal{C} \) is a collection of points in a manifold, with each coordinate parametrized by a parameter \( \lambda \). For a \( d \)-dimensional manifold, the curve will be, in some given coordinate system \( x^\mu \),

\[
\mathcal{C}(\lambda) = \{ (x^0, x^1, ..., x^d) | x^0 = x^0(\lambda), x^1 = x^1(\lambda), ..., x^d = x^d(\lambda) \}. \tag{2.38}
\]

We can define the tangent vector \( T^\mu \) of a curve by taking the derivative with respect to the parameter of the curve

\[
T^\mu = \frac{dx^\mu}{d\lambda} = \left( \frac{dx^0}{d\lambda}, \frac{dx^1}{d\lambda}, ..., \frac{dx^d}{d\lambda} \right). \tag{2.39}
\]

Many times when dealing with timelike paths, the parameter will be the proper time of the wordline, \( \lambda = \tau \), and the tangent vector will then be the four-velocity along the curve.

We classify vectors in the tangent spaces of a manifold into three groups: spacelike, null and timelike, by means of the sign of their norm. We define a vector \( V \) to be **spacelike** if

\[
V^\mu V_\mu > 0, \tag{2.40}
\]
and define it as a **null vector** if

\[ V^\mu V_\mu = 0, \tag{2.41} \]

and as a **timelike vector** if

\[ V^\mu V_\mu < 0. \tag{2.42} \]

This is valid also for tangent vectors at specific points of the curve. If a tangent vector satisfies to be in one of the above categories for all points along the curve, then the curve is said to be that kind of curve. Thus, a timelike curve is one which has a tangent vector which is timelike for all points on the curve. Only massless particles can travel on null curves, and all massive particles travel on timelike curves.

**Closed curves** are those curves for which the coordinates take the same value for two different values of the parameter. Finally, **closed timelike curves** (CTCs) are such curves which are closed and are timelike at all points. For proper time parametrized CTCs, the tangent vector will have norm -1 for all points.

The postulates of special relativity state that massless particles such as light travel with constant speed in all frames and is the upper bound of all possible speeds. In the report we will use units such that the speed of light \( c \), roughly \( c \approx 3 \times 10^8 \text{m/s} \), is set to \( c = 1 \), so that time will also have units of length and thus we can merge the time coordinate with the spatial coordinates if we let it be \( ct \), or in units \( c = 1 \), simply \( t \).

The fact that nothing can travel faster than light provides an easy way of portraying the possible events in a spacetime which can be reached by massive particles, in a **spacetime diagram**. Rigorously such a diagram is 4-dimensional in our physical world of three spatial dimensions and one temporal dimension, however to be able to view it pictorially, we suppress often two spatial dimensions and plot only one spatial coordinate to the temporal coordinate.

As an example, let us consider flat spacetime with the following **Minkowski metric** in Cartesian coordinates,

\[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{2.43} \]

and choose to suppress \( y \) and \( z \) coordinates, in a case where our worldline stays at constant \( y \) and \( z \), in which case the line element is

\[ ds^2 = -dt^2 + dx^2. \tag{2.44} \]

Light travels on null geodesics (with null tangent vector \( N \)), meaning that \( ds^2(N, N) = 0 \), satisfying

\[ \frac{dx}{dt} = \pm 1, \tag{2.45} \]

the two solutions being light traveling forward or backward in the \( x \)-direction with speed \( c = 1 \). In a \( x-t \)-spacetime diagram this would become straight lines through the origin, portrayed in Fig. 2 as yellow lines. When we add one additional spatial dimension to the diagram, the light rays will no longer be creating a perpendicular cross at the origin, but rather a cone, called **lightcones**. Adding the last spatial coordinate gives us the three-dimensional lightcone, but for simplicity we often portray it as the two-dimensional (surface) lightcone. These set the boundary for the possible travels for massive particles.
As stated earlier, massive particles travel only on timelike trajectories, so only between points which are timelike separated, meaning that if the separation is $\Delta x$, its norm should be negative,

$$ds^2(\Delta x, \Delta x) < 0 \Rightarrow 1 < \left(\frac{dt}{dx}\right)^2.$$  

This means that the slopes in the $x$-$t$-diagram are larger in absolute value than those of the light trajectories. Thus, timelike trajectories will confine themselves inside the lightcones, as seen in Fig. 2 between the point O and point T, two timelike separated events. Lightcones also determine the past and future of an event. All events in the lower half of the lightcone are the past and all above are the future points. In physical situations thus, a particle’s worldline is confined to the upper half of the lightcone, always traveling upwards in spacetime.

Similarly, if $ds^2(\Delta x, \Delta x) > 0$, then we are considering spacelike separated points, and the trajectory connecting them will be spacelike, as the trajectory between points O and S in Fig. 2.

From these definitions it is clear that each point has its own lightcone surrounding it, denoting all events which can be reached from the point through timelike paths. It is also clear that unless the lightcones tilt in the spacetime, one can never reach its own past, if following only timelike paths.
2.2 Einstein’s field equations

Einstein’s proposal in his theory of general relativity is that matter causes the geometry of the spacetime around it to curve, due to the matter’s gravitation. The geometry of the spacetime in turn however affects how particles travel in it (since the line element depends on the metric). The geometry of a spacetime is characterized by the metric, which in turn gives rise to the Riemann tensor, Ricci tensor and the Ricci scalar. Thus, these can all be used to characterize the curvature of a spacetime. This is exactly what happens in Einstein’s field equations (EFEs), where the connection between matter source and geometry is made. EFEs are a set of nonlinear second order differential equations for the metric.

In this report we present the principle of least action approach to recover EFEs, presented in more detail in Appendix A.

As in classical field theory, given a scalar Lagrangian density $L$, in $n$ dimensions, the action $S$ is given by

$$S = \int d^n x L.$$  \hspace{1cm} (2.46)

In general relativity, the Einstein-Hilbert action $S_H$ is the action built by setting the scalar to be the Ricci scalar together with the square root of the determinant of the metric $\sqrt{-g}$, which provides the volume element with the Lagrangian density to be tensorial,

$$S_H = \int d^n x \sqrt{-g} R = \int d^n x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}.$$  \hspace{1cm} (2.47)

This will describe the gravitational bit of the action, with no matter around. In general, there will also be an action due to the matter distribution, $S_M$, in the space and the total action is given by the sum of these contributions

$$S = S_H + S_M.$$  \hspace{1cm} (2.48)

By choosing appropriate normalizing factor in this sum of actions and by demanding the variation of the action with respect to the metric to vanish, and by imposing that this reduces to the Newtonian equations of gravity in the Newtonian weak-field limit (weak gravitational field meaning little curvature in the spacetime), we obtain Einstein’s field equations (see more exact derivation in Appendix A),

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$  \hspace{1cm} (2.49)

with $G$ being the gravitational constant in Newton’s gravitational law, and $G_{\mu\nu}$ is the already introduced Einstein tensor. In the equation, $T_{\mu\nu}$ is the stress-energy tensor of the matter source. It describes the pressure, energy, momentum, stress and strain of the spacetime. It is defined in several ways, all expressing the same information. In field theory, the Lagrangian density depending on a set of fields $\Phi_a$, we can express the stress-energy tensor as

$$T^{i}_{\nu} = \frac{\partial L}{\partial (\partial_i \Phi^j)} \partial_{\nu} \Phi^j - \delta^{i}_{\nu} L,$$ \hspace{1cm} (2.50)

or also by the definition given in Appendix A in (A.17), in terms of the matter action $S_M$.

By symmetry of (2.49), we can add a term with a constant $\Lambda$,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},$$ \hspace{1cm} (2.51)
which is still satisfied. This $\Lambda$ is called the cosmological constant. In the report’s discussion of closed timelike curves in solutions of EFEs, we will only regard spacetimes with $\Lambda = 0$.

Manifolds with a Ricci tensor proportional to the metric,

$$R_{\mu\nu} = cg_{\mu\nu}, \quad \text{where } c \text{ is a constant}, \quad (2.52)$$

are called Einstein manifolds or also Einstein spaces. All solutions to the vacuum EFEs (with completely vanishing stress-energy tensor) are Einstein spaces, since then,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

$$\Leftrightarrow R = d \left( \frac{1}{2} R - \Lambda \right)$$

$$\Leftrightarrow R_{\mu\nu} = g_{\mu\nu} \frac{2\Lambda}{d} \quad (2.53)$$

and inversely, if the Ricci tensor can be written in the form (2.53), then the EFEs will turn into vacuum EFEs. Einstein spaces will thus be any manifold with metric deduced from vacuum EFEs. As examples we have the Schwarzschild metric, the Kerr metric, and the background metric of the Gödel-type metrics, which we will encounter later.
2.3 Black hole solutions

2.3.1 Schwarzschild solution

Setting the cosmological constant $\Lambda = 0$ in EFEs and imposing a spherical symmetric and statical solution on the metric, and empty-space (vacuum) stress-energy tensor, we yield the Schwarzschild solution. In spherical coordinates $(t, r, \theta, \phi)$, spherical symmetry condition translates mathematically to $\partial_{\theta} g_{\mu \nu} = \partial_{\phi} g_{\mu \nu} = 0$ with $\mu, \nu = t, r$.

The line element at a fixed radial coordinate $r$ and a fixed time coordinate $t$, should reduce to the line element of a 2-sphere,

$$ds^2|_{(t,r)} = r^2 d\Omega^2. \quad (2.54)$$

The static condition translates to $\partial_t g_{\mu \nu} = 0$, for $\mu, \nu = t, r, \theta, \phi$, while the empty-space condition translates into $T_{\mu \nu} = 0$. Thus we yield EFEs as simply $G_{\mu \nu} = 0$.

This can be simplified to a condition on the Ricci tensor by contracting the Einstein tensor,

$$g^{\mu \nu} G_{\mu \nu} = R(1 - \frac{d}{2}) = 0 \quad \Leftrightarrow \quad R = 0, \quad (2.55)$$

which replacing back into the vacuum EFEs also implies $R_{\mu \nu} = 0$.

The static and spherical symmetric conditions and the imposition of leaving signature of Minkowski metric unchanged gives a first ansatz of the metric as

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2, \quad (2.56)$$

where $\alpha(r), \beta(r)$ are some functions of the coordinate $r$. Calculating the Riemann tensor and Ricci tensor (and scalar) in terms of these unknown functions in the metric, and imposing that they satisfy the given conditions, we receive solutions for the functions. Further, by imposing for the metric to be applicable to the Newtonian weak-gravitational field limit, one obtains the following complete solution for the Schwarzschild geometry,

$$ds^2 = -(1 - \frac{2GM}{r}) dt^2 + \frac{1}{(1 - \frac{2GM}{r})} dr^2 + r^2 d\Omega^2. \quad (2.57)$$

The remarkable about Schwarzschild geometry is the peculiar behavior of it at $r = 0$ and at $r = 2GM$. At these points, some components of the metric or the Ricci scalar diverge. Due to these behaviors we refer to the Schwarzschild geometry as a black hole in the cases where the radius of the object observed has a radius smaller than the $r = 2GM$ peculiarity. Calculating the Ricci scalar we find that

$$R = \frac{48G^2M^2}{r^6}. \quad (2.58)$$

which we see diverges at $r = 0$. Since the Ricci scalar is the same in all coordinate systems (since it is a scalar), this is a coordinate-independent peculiarity of the spacetime. This is called the *singularity* of the black hole. At $r = 2GM$, the scalar describing the spacetime does not diverge and so this point may be well-behaving in another coordinate system, where the components of the metric will be different.
Let us consider null radial rays \( (d\theta = d\phi = 0) \) in this geometry. For null curves with a parameter \( \lambda \), we have a vanishing line element, yielding us the equation from (2.57),

\[
0 = -(1 - \frac{2GM}{r}) dt^2 + \frac{1}{(1 - \frac{2GM}{r})} dr^2,
\]

and letting this direct product of basis one-forms act on a tangent vector \( V = \frac{dx^\mu}{d\lambda} \partial_\mu \) twice gives us

\[
0 = -(1 - \frac{2GM}{r}) (\frac{dt}{d\lambda})^2 + \frac{1}{(1 - \frac{2GM}{r})} (\frac{dr}{d\lambda})^2,
\]

\[
\Rightarrow \frac{dr}{dt} = \pm \frac{1}{(1 - \frac{2GM}{r})},
\]

from which we see that as \( r \to 2GM \), the lightcones tend to have an infinite inclination in the \((t, r)\) plane. This means that timelike trajectories, being confined to move inside lightcones, will never reach \( r = 2GM \). This is however a trick of the coordinate system: it is indeed possible to pass this \( r = 2GM \), called the event horizon of the black hole, but as seen from an observer in the coordinate system described, he will never see you pass it.

Most peculiar is however, that looking from the other side of the event horizon, for \( r < 2GM \), we see the similar thing happening: light never reaching outside the event horizon. This feature is however coordinate-independent: light can truly not escape a black hole once inside the event horizon. Seeing that the fastest possible velocity (null rays) can not escape this horizon, we can conclude that nothing can.

### 2.3.2 Kerr solution

If we still assume a vacuum solution to EFEs, but no longer impose spherical symmetry, but rather only axial symmetry (symmetry along one axis) and a solution which is stationary, but not necessarily static, then we get the Kerr solution. Kerr solution can again be regarded as a black hole. The metric is given in Boyer-Lindquist coordinates \((t, r, \theta, \phi)\) as

\[
ds^2 = -(1 - \frac{2GMr}{\rho^2}) dt^2 - \frac{2GMar}{\rho^2} \sin^2(\theta)(dt d\phi + d\phi dt) + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2(\theta)}{\rho^2} \sin^2(\theta) d\phi^2,
\]

with following definitions:

\[
a := \frac{J}{M},
\]

\[
\Delta := r^2 - 2GMr + a^2,
\]

\[
\rho^2 := r^2 + a^2 \cos^2(\theta).
\]

This black hole is a rotating black hole with angular momentum \( J \) around an axis. When \( J \to 0 \), this metric reduces to the Schwarzschild solution (2.57).

The Ricci scalar diverges at \( \rho = 0 \), which means that, from (2.62), \( r = 0, \theta = \frac{\pi}{2} \). However notice that in these coordinates, \( r \) is no longer the radial distance from the origin in the Cartesian coordinate
system, but rather the distance measured from a ring with radius \( a \) in the \( xy \)-plane, see Fig. (3) for geometry. Now the \( r = 0, \theta = \frac{\pi}{2} \) corresponds to the boundary of the disc portrayed in Fig. (3) and is now a ring singularity. Therefore, the coordinate \( r < 0 \) is defined and lies within this ring singularity. The spacetime inside the ring singularity is given simply by letting \( r < 0 \) in the metric of (2.61). We note however that there no longer is an event horizon in the spacetime inside the ring singularity.

Figure 3: Ellipsoidal geometry of the rotating black hole, the Kerr solution with a point \((r, \theta, \phi)\) outside the ring singularity. The coordinate \( r \) is measured from a ring singularity at \( r = 0 \) and \( \theta = \frac{\pi}{2} \). The angle \( \theta \) is measured from the vertical axis parallel to the \( z \)-axis through the point on the ring singularity from which \( r \) is measured to the point in question. Angle \( \phi \) is measured from the positive \( x \)-axis.
3 Classical closed timelike curves

Closed timelike curves occur in several exact solutions of EFEs. As defined, massive particles travel only on timelike trajectories, which also means that there is always proper time passing for them along the trajectory (since the line element does not vanish). Because the curve is closed, they will return to a point eventually which they have already passed: a spacetime point which has spatial as well as the temporal coordinate equal as a previous one. But then the traveler actually returns to a point backwards in time which it had already passed. This way, backwards time travel along CTCs is possible. This arouses many paradoxes in physics, which will be dealt with after we give the most famous examples of the occurrences of CTCs in exact solutions of EFEs.

For a closed timelike curve to appear we necessarily need the lightcone to tilt along the curve, as otherwise we could not have a timelike curve, since we could never cross the lightcones boundary to be able to come back to the initial point. Therefore, CTCs are usually portrayed as closed curves with lightcones following the curve in a sense that the curve is always inside a lightcone, see Fig. 4.

In this part we will examine the occurrence of CTCs in different exact solutions, only considering classical general relativity, without quantum mechanics. For each case we will discuss their potential existence in that spacetime and the problems they imply together with possible solutions to the problems. Finally we present some paradoxes for the existence of CTCs in general.

\[ ds^2 = \frac{(r_0^2 + a^2)^2 - a^2 \Delta}{\rho^2} d\phi^2. \] 

3.1 Kerr metric

3.1.1 Appearance of CTCs

Consider the family of curves \( C(\lambda) \) inside the ring singularity \( r = 0 \) in the Kerr metric, and at a constant time \( t = t_0 \), constant \( r = r_0 \) with \( |r_0| \ll 1 \) and \( r_0 < 0 \) and at constant azimuthal angle \( \theta = \frac{\pi}{2} \) (so that \( \sin(\theta) = 1 \)) for convenience of calculations. The line element along this curve, given by (2.61) with \( dt = dr = d\theta = 0 \) is

\[ ds^2 = \frac{(r_0^2 + a^2)^2 - a^2 \Delta}{\rho^2} d\phi^2. \]
Using the values for the notational functions according to (2.62), we can write

$$ds^2 = \frac{(r_0^2 + a^2)^2}{r_0^2} - a^2(r_0^2 - 2GMr_0 + a^2)d\phi^2 =$$

$$= \frac{1}{r_0^2}(r_0^4 + 2a^2r_0^2 + a^4 - a^2r_0^2 + 2GMra^2 - a^4)d\phi^2 =$$

$$= \left( r_0^2 + a^2 + \frac{2GMA^2}{r_0} \right) d\phi^2. \quad (3.2)$$

Now using that $|r_0| \ll 1$,

$$ds^2 \approx \left( a^2 + \frac{2GMA^2}{r_0} \right) d\phi^2 < 0, \quad (3.3)$$

which is strictly negative for $r_0 < 0$ (and small $r_0$). Therefore, our curves $C(\lambda) = x^\mu(\lambda)$ with tangent vector

$$t^\mu := \frac{dx^\mu}{d\lambda} = \left( 0, 0, 0, \frac{d\phi}{d\lambda} \right), \quad (3.4)$$

will be timelike for all points, as the norm of the tangent vector

$$t_\mu t^\mu = g_{\mu\nu}t^\mu t^{\nu} = g_{33} \left( \frac{d\phi}{d\lambda} \right)^2 < 0, \quad (3.5)$$

and thus the curves will be timelike. We also note that the points in this spacetime for which $\phi = 0$ and $\phi = 2\pi$ are identified, due to the choice of these coordinates (it is a full revolution around the symmetry axis), so it is also closed, yielding us our first example of CTCs occuring in the inside of the ring singularity of the rotating black hole.

### 3.1.2 Possible problems

There are few problems arising with this occurrence of CTCs, but least of all the spacetimes we consider in the report. One questionable block is whether we can consider the angular coordinate $\phi$ to still be identified when it becomes timelike as it is when being spacelike. This aspect arises also in the Gödel universe and will be discussed in more depth in section 3.2.2.

The best feature of this spacetime is its asymptotic flatness. At large distances from the rotating black hole and the spatial location of the CTC, the spacetime is almost flat. Far from a point in a spacetime where there is matter field or where the geometry becomes more irregular, we assume the universe to not have any curvatures or changes from a flat spacetime. Therefore, any spacetime possessing asymptotic flatness is argued to be more physical. CTCs in the rotating black holes are thus in a sense the most physical and realistic appearance of these curves. However one can question how possible it is practically to reach a point inside the ring singularity, but the practicalities are not discussed in this report.
3.2 Gödel universe and Gödel-type metrics

3.2.1 Appearance of CTCs

The Gödel universe was derived in 1935 by Kurt Gödel. In later research, a whole group of metrics were developed, called the Gödel-type metrics, which possess some unique characteristics. By definition, Gödel-type metrics are those spacetimes where the metric \( g_{\mu\nu} \) can be written as the difference between a degenerate background metric \( b_{\mu\nu} \) of an Einstein space of one dimension lower than the entire spacetime and the tensor product of two unit timelike vectors \( u_{\mu} \) [6],

\[
g_{\mu\nu} = b_{\mu\nu} - u_{\mu}u_{\nu}, \tag{3.6}
\]

meaning that the background metric \( b_{\mu\nu} \) in matrix form is a \( d \times d \) dimensional matrix with a zero row (therefore a matrix with rank \( d - 1 \)). As an example, let us consider the trivial case of the Gödel universe and how this is a Gödel-type spacetime.

The Gödel universe is a solution to EFEs for an infinitely long rotating dust cylinder (a pressureless perfect fluid source rotating around an axis). The metric is given by [9]

\[
ds^2 = -dt^2 + dx^2 - \frac{1}{2}e^{2x}dy^2 + dz^2 - e^x (dtdy + dydt), \tag{3.7}
\]

in a coordinate system \((t, x, y, z)\) which not necessarily denotes Cartesian coordinates for the spatial part. The metric in matrix form is then

\[
g_{\mu\nu} = \begin{bmatrix} -1 & 0 & \frac{-e^x}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -e^x & 0 & \frac{1}{2}e^{2x} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

with an inverse metric of

\[
g^{\mu\nu} = \begin{bmatrix} 1 & 0 & -2e^{-x} & 0 \\ 0 & 1 & 0 & 0 \\ -2e^{-x} & 0 & 2e^{-2x} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

If we let an index-lowered four-velocity \( u_{\mu} \) be:

\[
u_{\mu} = (1, 0, e^x, 0), \tag{3.8}
\]

yielding a \((0, 2)\)-type tensor of its tensor product with itself

\[
u_{\mu}u_{\nu} = \begin{bmatrix} 1 & 0 & e^x & 0 \\ 0 & 1 & 0 & 0 \\ e^x & 0 & e^{2x} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

and writing a background metric \( b_{\mu\nu} \) as


\[ b_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]

then indeed, we can write the Gödel universe metric as

\[ g_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 e^{2x} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & e^x & 0 \\ 0 & 1 & 0 & 0 \\ e^x & 0 & e^{2x} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = b_{\mu\nu} - u_{\mu} u_{\nu}. \]

Note that \( b_{\mu\nu} \) is indeed a degenerate metric of dimension \( d = 4 \) and rank \( d - 1 = 3 \) (our spacetime is of dimension \( d = 4 \)). Note also that the norm of the four-velocity is

\[ u_{\mu} u^{\mu} = g^{\mu\nu} u_{\mu} u_{\nu} = g^{00} u_{0} u_{0} + 2 g^{20} u_{0} u_{2} + g^{22} u_{2} u_{2} + g^{33} u_{3} u_{3} = 1^2 + 2(-2 e^{-x}) e^x + 2 e^{-2x} (e^x)^2 = -1, \]

so indeed, this vector is a timelike unit vector. We have thus written the Gödel universe in the Gödel-type metric form. Note however that this is not a unique rewriting: we can choose some other degenerate background metric and another timelike unit vector as well, another example is given by

\[ b_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]

and a timelike unit vector (with lowered index),

\[ u_{\mu} = (\sqrt{2}, 0, \frac{1}{\sqrt{2}} e^x, 0), \]

where the background metric now is actually a flat, Euclidean degenerate metric.

Now let us examine closed timelike curves in the Gödel universe. We do this by going to another set of coordinates \((t, r, \phi, z)\) through a cylindrical transformation, yielding the line element on the form

\[ ds^2 = -dt^2 + dr^2 + dz^2 - \sinh^2 r (\sinh^2 r - 1) d\phi^2 + \sqrt{2} \sinh^2 r (d\phi dt + dt d\phi), \]

with ranges

\[ t \in (-\infty, \infty), \quad r \in (0, \infty), \quad z \in (-\infty, \infty), \quad \phi \in [0, 2\pi], \]

with \( \phi = 0 \) and \( \phi = 2\pi \) being identified and being the point where the curve closes.

Firstly, let us consider a curve \( \gamma(s) \) with all coordinates constant (with \( r = r_C \)), except for the angular coordinate \( \phi \), parametrized by a parameter \( s \). Its tangent vector then reads

\[ V^\mu := \frac{d\gamma^\mu}{ds} = (0, 0, \frac{d\phi}{ds}, 0). \]
For $\gamma$ to be a timelike curve, we demand the tangent vector to be timelike (and normalized as a four-velocity) for all points $s$ on the curve. This we receive by writing the line element along the curve, with $\,dt = dr = dz = 0$,
\[
ds^2 = -\sinh^2 r_C (\sinh^2 r_C - 1) d\phi^2,
\]  
where $r_C$ is the constant radial coordinate at which the curve is positioned. This yields the equation of demand of $V$ being timelike, for some set of radial coordinates $r_C$
\[
V^\mu V^\nu < 0,
\]  
which becomes
\[
g_{\mu\nu} V^\mu V^\nu = -\sinh^2 r_C (\sinh^2 r_C - 1) \left( \frac{d\phi}{ds} \right)^2 < 0
\Rightarrow \sinh^2 r_C - 1 > 0,
\]  
solving this hyperbolic equation gives us:
\[
r_C > \log(1 + \sqrt{2}).
\]

Figure 5: Gödel universe: a cylindrically symmetric rotating dust matter distribution. The tilting of the lightcones for radii larger than $r_c$ is portrayed for curves with constant $t, r, z$. The figure shows the spacetime at a time slice at some constant $t$. Radius $r$ is measured from the axis of rotation, the $z$-axis.

Which means that whenever we are outside the radius $r_C$, then we will be traveling on a CTC which is constant in coordinate time, see Fig. 5 for the tilting of the light cones for radii larger than $r_c$. However, during our travel some proper time elapses, and when returning to the initial point $\phi = 0$, we have traveled back to the point of departure, which for us will be a time travel backwards in time. The amount of proper time $\Delta \tau$ passing for the traveler will be given from the line element on the curve (3.14) and the definition of proper time
\[
\int d\tau^2 = -ds^2 = \sinh^2 r_C (\sinh^2 r_C - 1) d\phi^2,
\Rightarrow \Delta \tau = \int_0^{2\pi} \sinh r_C \sqrt{\sinh^2 r_C - 1} \, d\phi = 2\pi \sinh r_C \sqrt{\sinh^2 r_C - 1},
\]  
which will be defined and positive once we have the condition (3.16) for a timelike curve.
Let us return to the general Gödel-type metrics. We can always do a cylindrical coordinate change in these metrics, [6], so that the line element becomes
\[ ds^2 = dr^2 + r^2d\phi^2 + dz^2 - (dt + s(r, \phi)dz)^2. \]
(3.19)
with a metric on matrix form:
\[ g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & -s \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ -s & 0 & 0 & 1 - s^2 \end{pmatrix}, \]
with \( s(r, \phi) \) being an arbitrary function of the coordinates \( r \) and \( \phi \). Let us now consider a general curve \( \gamma(\lambda) \), with all coordinates having a general dependence on the parameter, where our parameter \( \lambda \in [0, 2\pi] \), which we can always compactify our parameter to. The tangent vector \( T^\mu \) to the curve is
\[ T^\mu = \frac{d\gamma}{d\lambda} = \left( \frac{dt}{d\lambda}, \frac{dr}{d\lambda}, \frac{d\phi}{d\lambda}, \frac{dz}{d\lambda} \right). \]
(3.20)
Now impose for this curve to be timelike (normalized), and use the form of the metric (3.19) to explicitly calculate it,
\[ V_\mu V^\mu = -1, \]
which gives us
\[ g_{00}V^0V^0 + 2g_{03}V^0V^3 + g_{11}V^1V^1 + g_{22}V^2V^2 + g_{33}V^3V^3 = -1 \]
\[ \Leftrightarrow -\left( \frac{dt}{d\lambda} \right)^2 - 2\frac{dt}{d\lambda}\frac{dz}{d\lambda} + \left( \frac{dr}{d\lambda} \right)^2 + r^2\left( \frac{d\phi}{d\lambda} \right)^2 + (1 - s^2)\left( \frac{dz}{d\lambda} \right)^2 = -1 \]
\[ \Rightarrow \frac{dt}{d\lambda} = -\frac{dz}{d\lambda} \pm \sqrt{s^2\left( \frac{dz}{d\lambda} \right)^2 + \left( \frac{dr}{d\lambda} \right)^2 + r^2\left( \frac{d\phi}{d\lambda} \right)^2 + (1 - s^2)\left( \frac{dz}{d\lambda} \right)^2 + 1} \]
\[ \Leftrightarrow \frac{dt}{d\lambda} = -\frac{dz}{d\lambda} \pm \sqrt{\left( \frac{dr}{d\lambda} \right)^2 + r^2\left( \frac{d\phi}{d\lambda} \right)^2 + \left( \frac{dz}{d\lambda} \right)^2 + 1}. \]
(3.21)
Let us now Fourier-series expand this expression conventionally, having \( \lambda \) range in the entire defined interval \([0, 2\pi]\). The functions \( r(\lambda), \phi(\lambda), z(\lambda) \) are assumed to be periodic in \( \lambda \) at this stage, in order to have CTCs. Fourier-expansion is
\[ \frac{dt}{d\lambda} = \sum_{n=-\infty}^{\infty} c_n e^{in\lambda}, \]
(3.22)
with the coefficients given by
\[ c_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{d\lambda} e^{-in\lambda} d\lambda. \]
(3.23)
Now look at the first term in the expansion, \( n = 0 \) with coefficient \( c_0 \), which will be the constant term.
in our expansion. From the expression (3.21) it reads

\[
c_0 = \frac{1}{2\pi} \int_0^{2\pi} \left( -\frac{dz}{d\lambda} \pm \sqrt{\left( \frac{dr}{d\lambda} \right)^2 + r^2 \left( \frac{d\phi}{d\lambda} \right)^2 + \left( \frac{dz}{d\lambda} \right)^2 + 1} \right) d\lambda =
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left( -\frac{dz}{d\lambda} \right) d\lambda \pm \frac{1}{2\pi} \int_0^{2\pi} \left( \sqrt{\left( \frac{dr}{d\lambda} \right)^2 + r^2 \left( \frac{d\phi}{d\lambda} \right)^2 + \left( \frac{dz}{d\lambda} \right)^2 + 1} \right) d\lambda =
\]

\[
=: \ c_{0a} \pm c_{0b} \cdot \tag{3.24}
\]

However notice that the integrand of the second integral is positive definite, \( c_{0b} > 0 \). After plugging this in (3.21) and integrating, these constants will give a linear term

\[
t(\lambda) = (c_{0a} \pm c_{0b})(\lambda - \lambda_0) + \frac{1}{2\pi} \int_{\lambda_0}^{\lambda} \sum_{n \neq 0} c_n e^{in\lambda} d\lambda. \tag{3.25}
\]

Assuming, as stated earlier, if all other components are periodic in \( \lambda \) (except for \( t \) for now), then we can find solutions such that the integral term in this (3.25) is periodic in \( \lambda \). However for \( t \) to be periodic, since \( \lambda \neq \lambda_0 \), we must have that \( c_{0a} \pm c_{0b} \) vanishes. But we already know that \( c_{0b} \) is strictly positive definite. Therefore we can conclude that we can not have CTCs if \( c_{0a} = 0 \), for then the linear term in \( \lambda \) has no chance of vanishing. Now let us look at what this condition \( c_{0a} = 0 \),

\[
\frac{1}{2\pi} \int_0^{2\pi} \left( -s(r, \phi) \frac{dz}{d\lambda} \right) d\lambda = 0, \tag{3.26}
\]

where we now added the explicit dependence of the function \( s(r, \phi) \). Since we are assuming \( r(\lambda) \) and \( \phi(\lambda) \) to be periodic in order to find CTCs, we also have that \( s(r, \phi) \) is periodic, which lets us Fourier-series expand it

\[
s(r(\lambda), \phi(\lambda)) = d_0 + f(\lambda) = d_0 + \sum_{n \neq 0} d_n e^{in\lambda}, \tag{3.27}
\]

where \( f(\lambda) \) is guaranteed to be a periodic function. Now replacing this in our expression (3.26) we get

\[
-\frac{1}{2\pi} d_0 \int_0^{2\pi} \frac{dz}{d\lambda} d\lambda - \frac{1}{2\pi} \int_0^{2\pi} f(\lambda) \frac{dz}{d\lambda} d\lambda = 0. \tag{3.28}
\]

Since we are assuming \( z(\lambda) \) to be periodic function, the first integral will vanish, leaving us with the demand of

\[
\int_0^{2\pi} f(\lambda) \frac{dz}{d\lambda} d\lambda = 0. \tag{3.29}
\]

But in a search of CTCs we have the freedom to choose the periodic function \( z(\lambda) \) as we wish, say exactly so that its derivative is \( f(\lambda) \), and so we get

\[
\int_0^{2\pi} f(\lambda)^2 d\lambda = 0. \tag{3.30}
\]

This is only possible if \( f(\lambda) \) is constant, and thus also \( s(r, \phi) \) is a constant. However, suppose we exclude the case of constant \( s(r, \phi) \), then we can always find periodic functions \( s(r, \phi), z(\lambda), r(\lambda), \phi(\lambda) \)

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such that indeed the demand (3.26) is not satisfied, and then we can choose the coefficients $c_{0a}$ such that it cancels $c_{0b}$ and so $t(\lambda)$ can also be periodic.

We have rephrased our question about whether CTCs can exist in this metric to the question whether $s(r, \phi)$ is constant or not. If this function is not constant, then there always exist closed timelike curves in the Gödel-type metrics.

### 3.2.2 Possible problems

If we derive the geodesics of the Gödel-type metrics, we note that these CTCs are not geodesics of the spacetime. This implies that a massive particle will need an external force acting on it and so forcing it to move on the given closed timelike curve. The interaction connecting the particle and this forcing machine will need to essentially also travel on a closed timelike curve to be able to complete the entire trajectory. But then this forcing machine, also being a massive particle by assumption, needs also something to force it on a CTC, and would that be the particle it is initially forcing? This would mean that they are producing an internal force which pushes the two together, but we need an external force acting on the entire system as a whole. What will push the system of the particle+force machine? We need a third object pushing the two, and then we are back in the beginning and we can go on forever. To solve this paradox we need a force machine which can act over time and space and needs not the interaction to follow the same type of path as the forced particle, however this is an unphysical interaction which needs more study.

In our derivations of the CTCs in the Gödel-type metrics, we were assuming all the time the identification of the angular coordinate $\theta = 0$ with $\theta = 2\pi$, even for the case (3.14), when the angular coordinate $\phi$ becomes a timelike coordinate. There is no mathematical or physical demand which allows us to still identify the angular coordinate as we do in the case when it is spacelike as in the region where it becomes timelike. However, there is also no demand that we should not be able to identify them. To portray why the identification can cause confusion among physicists, let us consider Minkowski space in the usual cylindrical coordinates [10],

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2,$$

where we are identifying points $\theta = 0$ with $\theta = 2\pi$. Consider now a coordinate transformation ($\alpha$ being a constant)

$$t' = t + \alpha \theta, \quad dt = dt' - \alpha d\theta,$$
$$r' = r, \quad dr = dr',$$
$$\theta' = \theta, \quad d\theta = d\theta',$$
$$z' = z, \quad dz = dz',$$

upon which the metric becomes

$$ds^2 = -(dt' - \alpha d\theta)^2 + dr'^2 + r'^2 d\theta'^2 + dz'^2 =$$
$$= -dt'^2 + \alpha(dt' d\theta' + d\theta' dt') + dr'^2 + (r'^2 - \alpha^2) d\theta'^2 + dz'^2. \quad (3.32)$$

The angular coordinate $\theta'$ becomes timelike for $r' < \alpha$ if we consider the curve

$$C = \{t' = \text{constant}, r' = \text{constant}, z' = \text{constant}, \theta' = \theta'(\lambda)\},$$

(3.33)
which has a timelike tangent vector at all points. Imposing further that \( \theta' = 0 \) and \( \theta' = 2\pi \) be identified even in the new coordinates would yield \( C \) to be a closed timelike curve. But choosing \( \alpha \) arbitrarily large we could have CTCs practically in the entire spacetime of our approximately flat Minkowski space. This is however not the case, or at least we are unable to detect them. Perhaps they form a black hole before they are able to be used as paths (see discussion in 8.2).

Perhaps the reason we are finding CTCs in Minkowski space is the fact that we are imposing an identification to be made for the new angular coordinates even after they cease being spacelike. Maybe this identification is not physical and there is some (yet) unknown physical explanation to why this identification can not be made. Forbidding this kind of identification after coordinate transformation would solve the existence of CTCs in the rotating black hole, the Gödel universe and Minkowski spacetime.

### 3.3 Gott’s cosmic strings

#### 3.3.1 Appearance of CTCs

A cosmic string is a cylinder with cylindrical symmetry inside and outside, for which the radius of the cylinder is taken to be infinitely small. Thus, to examine these strings, we first examine spacetimes exhibiting cylindrical symmetry which are also static (no time dependence in the metric). The most general form of this is given in [12]

\[
ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + e^{2\Psi(r)}d\theta^2 + e^{2\lambda(r)}dz^2,
\]  

(3.34)

with \( \nu(r) \), \( \lambda(r) \), \( \Psi(r) \) are some functions of the \( r \) coordinate. From this general form, having a stress-energy tensor (a matter source in the spacetime), we can derive the exact expressions of these unknown functions.

In field theory, a Lagrangian density \( \mathcal{L} \), depending on fields \( \Phi^i(x) \), describes the matter distribution in the spacetime. The stress-energy tensor is defined in terms of these fields as (given in general quantum field theory literature or in [14])

\[
T^\mu_{\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^i)} \partial_\nu \Phi^i - \delta^\mu_\nu \mathcal{L},
\]  

(3.35)

where we sum over all fields \( i = 1, ..., N \), where \( N \) is the number of fields in our theory.

Let us now place an infinite string along the \( z \)-axis, without loss of generality. The string has radius \( r_0 \) and the spacetime is divided into two regions: the interior region, \( r < r_0 \), inside the string, and the exterior region, \( r > r_0 \), outside the string. Let this setup be so that the string is invariant under translations in the \( z \)-direction (the reason why we are considering it to be infinite), so that the fields describing our theory will only be dependent on the \( x \)- and \( y \)-coordinates. According to the definition of the stress-energy tensor in (3.35), the diagonal components in the first and last index will be the same, since we have then that \( \partial_\mu \Phi^i = 0 \) for \( \mu = t, z \), and

\[
T^t_t = T^z_z = -\mathcal{L} =: -\kappa,
\]  

(3.36)

where we defined \( \kappa \) to be just as given, as the energy density of the string. For a uniform string, this will be a constant. With help of the stress-energy conservation law \( \nabla_\mu T^\mu_{\nu} = 0 \), we can deduce that all other elements of the stress-energy tensor are constant, but we also demand these to vanish at large
distances from the string (since we have pure vacuum far from the string). In the end then, we are left with a stress-energy tensor for the interior region, with only two non-zero components,

\[ T_{\mu \nu} = -\kappa \text{diag}(1, 0, 0, 1). \]  

The four independent Einstein’s field equations are, for the diagonal components (see Appendix B for exact derivations)

\[ e^{-2\lambda} [\Psi'' + \Psi'^2 + \lambda''] = -8\pi G \kappa, \]  

\[ e^{-2\lambda} [\nu' \Phi' + \nu' \lambda' + \Psi' \nu'] = 0, \]  

\[ e^{-2\lambda} [\nu'' + \nu'^2 + \lambda''] = 0, \]  

\[ e^{-2\lambda} [\nu'' + \nu'^2 - \nu' \lambda' + \nu' \Psi' - \lambda' \Psi' + \Psi'' + \Psi'^2] = -8\pi G \kappa. \]  

Now we apply the conservation law for the stress-energy tensor

\[ \nabla_{\mu} T_{\mu \nu} = 0, \]  

for \( \nu = r \), we get, by the definition of the covariant derivative on a \( (1, 1) \)-type tensor (we find the Christoffel symbols used here in Appendix B)

\[ \nabla_{\mu} T_{\mu r} = \partial_{\mu} T_{\mu r} + \Gamma_{\mu \lambda} T_{\lambda r} = -\nabla_{\mu} T_{\mu r} = -\nu'(-\kappa) - \lambda'(-\kappa) = \kappa(\nu' + \lambda') = 0. \]  

For a uniform string with constant \( \kappa \), we get solutions \( \nu' = -\lambda' \). Using this in the second EFE (3.39), we get

\[ \nu' \Psi' - \nu'^2 - \nu' \Psi = 0, \]  

\[ \Rightarrow \nu' = 0, \]  

\[ \Rightarrow \lambda' = 0, \]  

so the two functions are constant. We can rescale our coordinates \( (t, r, \theta, z) \) so that in the cylindrical metric, the components will be \( \pm 1 \) for all except for the angular one-form basis, so that \( \nu = \lambda = 0 \). Having said all this, both EFEs (3.38) and (3.40) reduce to

\[ \Psi'' + \Psi'^2 = -8\pi G \kappa. \]  

This differential equation can be solved by the substitution \( e^{\Psi(r)} = W(r) \), yielding a trigonometric solution

\[ W(r) = C \cos(\sqrt{8\pi G \kappa} r) + D \sin(\sqrt{8\pi G \kappa} r). \]  

At the middle of the string at \( r = 0 \), we need the metric to be nondegenerate, meaning that the angular coordinate vanishes at this point, otherwise we would have a circular coordinate for a zero radius, giving us degeneracies. We can also scale the function \( W(r) \) so that the derivative at \( r = 0 \) be equal to 1. Thus, we are left with the \( g_{\theta \theta} \) component of the metric being

\[ g_{\theta \theta} = W^2 = \frac{1}{8\pi G \kappa} \sin(\sqrt{8\pi G \kappa} r), \]  

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and together with the rescaling of the coordinates as already mentioned, we get the interior solution of the string to be

\[
\begin{align*}
    ds^2 &= -dt^2 + dr^2 + dz^2 + \frac{1}{8\pi G\kappa} \sin^2(\sqrt{8\pi G\kappa}r) d\theta^2. 
\end{align*}
\]  

(3.48)

Now we also need a solution to the exterior region around the string, where we assume vacuum to be the matter source. Cylindrical symmetric, static spacetimes with vacuum solutions are solved in [11] generally. Our wish is for the spacetime to have no extrinsic pressure at the boundary of the string. This means that at the boundary \( r = r_0 \), we want the interior and the exterior solutions to match. With these impostures, the exterior solution is solved to be

\[
\begin{align*}
    ds^2 &= -dt^2 + dz^2 + dr^2 + (1 - 4\mu)^2 r^2 d\theta^2. 
\end{align*}
\]  

(3.49)

Here, \( \mu \) is the linear energy density of the string and is obtained by integrating the energy density over a horizontal slice of a string (at a constant \( z \)) and at a slice in time (at a constant \( t \)),

\[
\mu = \int \kappa \sqrt{\gamma} |drd\theta|,
\]  

(3.50)

where \( \gamma \) is the determinant of the induced metric \( \gamma_{\mu\nu} \) on the surface of integration. We are considering \( z = \text{constant}, t = \text{constant} \), so our line element for the interior solution becomes

\[
\begin{align*}
    \gamma_{\alpha\beta} dx^\alpha dx^\beta &= dr^2 + \frac{1}{8\pi G\kappa} \sin^2(\sqrt{8\pi G\kappa}) d\theta^2. 
\end{align*}
\]  

(3.51)

This gives a determinant for the \( 2 \times 2 \) matrix \( \gamma_{\alpha\beta} \)

\[
\gamma = \frac{1}{8\pi G\kappa} \sin^2(\sqrt{8\pi G\kappa}),
\]  

(3.52)

yielding the integral

\[
\begin{align*}
    \mu &= \int_0^{2\pi} \int_0^{r_0} \kappa \frac{1}{\sqrt{8\pi G\kappa}} \sin(r \sqrt{8\pi G\kappa}) dr d\theta = \\
    &= 2\pi \kappa \frac{1}{\sqrt{8\pi G\kappa}} \cos(r \sqrt{8\pi G\kappa}) |_0^{r_0} = \\
    &= 2\pi \kappa \frac{1}{\sqrt{8\pi G\kappa}} \frac{1}{\sqrt{8\pi G\kappa}} (1 - \cos(r_0 \sqrt{8\pi G\kappa})) = \\
    &= \left(-\frac{1}{4G}\right) (1 - \cos(r_0 \sqrt{8\pi G\kappa})). 
\end{align*}
\]  

(3.53)

For our purposes, we will consider CTCs appearing in the vacuum surrounding the cosmic string and will thus be interested in the exterior solution (3.49). First, consider a coordinate transformation leaving all coordinates but the angular one to change according to

\[
\theta \rightarrow \theta' = (1 - 4\mu) \theta, \quad \Rightarrow \quad d\theta' = (1 - 4\mu) d\theta. 
\]  

(3.54)

Yielding us the metric to be

\[
\begin{align*}
    ds^2 &= -dt^2 + dz^2 + dr^2 + r^2 d\theta'^2. 
\end{align*}
\]  

(3.55)
and the identification of \( \theta = 0 \) with \( \theta = 2\pi \) will now be identification of \( \theta' = 0 \) with \( \theta' = 2\pi (1 - 4\mu) \).

For convenience let us rename \( \theta \to \theta' \). We see that the metric (3.55) is simply Minkowski space in the usual cylindrical coordinates, with the exception that our angular coordinate does not cover the entire revolution in the \( xy \)-plane, but will have a wedge in the \( xy \)-plane. There will be an angle \( D \) missing in the plane, since we are saying that \( \theta = 2\pi (1 - 4\mu) = 2\pi - 8\pi\mu \) is coming back to the starting point \( \theta = 0 \), so we are ending \( D = 8\pi\mu \) before the normal coordinates would identify itself. This \( D \) will be our angle deficit of the spacetime. This wedged Minkowski space can be rolled up into a conic spacetime, portrayed in Fig. 6, where the right figure is a segment of flat Minkowski space with an angle \( D \) missing, and gluing the yellow sides together yields the left conic spacetime (with the \( t \) and \( z \) coordinates suppressed).

For convenience again, let us go to a Cartesian coordinate system with the usual identification of radial and angular components as:

\[
\begin{align*}
t &= t, \\
x &= r\cos(\theta + 4\pi\mu), \\
y &= r\sin(\theta + 4\pi\mu) + d, \\
z &= z,
\end{align*}
\]

(3.56)

upon which the metric becomes the usual Minkowski metric in Cartesian coordinates

\[
ds^2 = -dt^2 + dx^2 + dy^2 + dz^2,
\]

(3.57)

where we have moved the \( y \)-coordinate with a distance \( d \) above the origin. We identify the points on the sides of the wedge. From simple trigonometry that can be used in flat Minkowski space, we find that the \( x \)-coordinate is

\[
x = (y - d)\tan(\theta + 4\pi\mu),
\]

(3.58)

and we identify \( \theta = 0 \) with \( \theta = 2\pi - 8\pi\mu \) (at constant \( y \)), so the two \( x \)-coordinates at these two angles are identified, \( x_0 = x_{2\pi} \), with \( x_0 = (y - d)\tan(4\pi\mu) \), giving us

\[
x_{2\pi} = (y - d)\tan(2\pi - 8\pi\mu + 4\pi\mu) = (y - d)\tan(2\pi - 4\pi\mu) =
\]

\[
= (y - d)\tan(-4\pi\mu) = -(y - d)\tan(4\pi\mu) = -x_0,
\]

(3.59)

thus we have the identification of \( x = -x \) (if \( y \) is constant). Now let us consider two parallel cosmic strings, located at a distance \( 2d \) from each other, running in the \( z \)-axis direction. Two observers A and B are located at opposite sides of the strings, see Fig. 7. The spacetime corresponding to the situation patches together two wedged Minkowski spaces, so that there is a wedge at \( y < 0 \) and \( y > 0 \) with total deficit angle (the angle of the wedge in flat space) \( D = 2\theta \), see Fig. 8.

Observer A sends light signals to B. If there was one cone between A and B, then B would see two light rays with an angle \( D \) between, according to Fig. 6. Now in the case of two cones, the light ray of the cones passing the side closer to the other cone will merge with the other ray, so we have only three paths the light takes, path 1, 2 and 3 as shown in Fig. 8.

By Pythagoras theorem in the flat Euclidean space of the folded out cone, we have the relation:

\[
(d + c\cos(\theta))^2 + (b - c\sin(\theta))^2 = a^2,
\]

(3.60)

with the sides \( a, b, c, d \) defined as in Fig. 8. Due to a gravitational lensing effect described in [13], light rays traveling on paths 2 and 3 will arrive to observer B earlier than along path 1. The path a which
Figure 6: A conic spacetime (left) with angle deficit $D$. A light source (yellow) emits light at one side of the cone. The observer (grey) receives two light rays. Conic spacetime flattened out (right) shows the simple Euclidean trigonometric relations, showing that $D = 2\beta$.

Figure 7: Two strings running parallel at distance $2d$ and two observers $A$ and $B$ located at two opposite sides inbetween the strings.

the light travels is received by minimizing (3.60) with respect to $c$ (since light always takes the shortest path), demanding $\frac{da}{dc} = 0$, which is reached at

$$c = b\sin(\theta) - d\cos(\theta), \quad (3.61)$$

upon which $a$ becomes, after replacing this expression in (3.60),

$$a^2 = b^2 + d^2 - c^2, \quad (3.62)$$

so if $c > d$ then $b > a$, and so $a$ will be shorter than $b$. In units where the speed of light $c = 1$, the time to travel along path 1 is $\Delta t_1 = 2b$, and along 2 (or 3) is $\Delta t_{2,3} = 2a$, giving a time delay between the two different signals of $\Delta t_{\text{signal}} = 2(b - a)$.

Since light is an upper limit of speed, we could originally not have a timelike trajectory reaching a point on the same path faster than light. But in this case light reaches a point faster than another light ray taking another path, because it is a shorter path. However since we are considering a path which is shorter and faster for light, we can always choose a particle moving fast enough close to the limit
of speed which also travels faster on the shorter path 2 or 3 than the other light along path 1. So we can have a space machine traveling at a speed \( v < c \) on a timelike trajectory through path 2 or 3 from observer A to B, reaching it before the light going through path 1.

Now let us consider the events \( e_1 \) and \( e_2 \) which are identified, since they are lying on the edge of the wedge, along path 2, see Fig. 8, just as events \( e_3 \) and \( e_4 \) lying on the lower wedge edge on path 3. In the Cartesian coordinate system defined in (3.56), we can express events as, if we set all to happen at an origin of the time-coordinate for convenience, in order \((t, x, y, z)\),

\[
\begin{align*}
e_1 &= (0, x_0, y_0, 0), \\
e_2 &= (0, -x_0, y_0, 0), \\
e_3 &= (0, -x_0, -y_0, 0), \\
e_4 &= (0, x_0, -y_0, 0),
\end{align*}
\]

(3.63)

where

\[
\begin{align*}
x_0 &= c\sin(\theta), \\
y_0 &= c\cos(\theta) + d,
\end{align*}
\]

(3.64)

by the simple geometry of the situation.

Now we boost the two strings in opposite directions along the \( x \)-axis with speed \( v_b \), corresponding to a boost matrix
But then we might as well choose $d$, but we had that the space machine is a massive object traveling on a timelike trajectory, so $v$ as we stated earlier directly after equation (3.62), yielding

$$B_{\pm} = \begin{bmatrix}
\gamma_b & \pm v_b \gamma_b & 0 & 0 \\
\pm v_b \gamma_b & \gamma_b & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},$$

where the $B_+$ boosts the upper string, so all $y > 0$, and $B_-$ boosts the lower string, all $y < 0$. The events $e_i$ will then become

$$e_1' = e_1 B_+ = (0, x_0, y_0, 0) \begin{bmatrix}
\gamma_b & v_b \gamma_b & 0 & 0 \\
v_b \gamma_b & \gamma_b & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = (x_0 v_b \gamma_b, x_0 \gamma_b, y_0, 0)$$

$$e_2' = e_2 B_+ = (0, -x_0, y_0, 0) \begin{bmatrix}
\gamma_b & v_b \gamma_b & 0 & 0 \\
v_b \gamma_b & \gamma_b & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = (-x_0 v_b \gamma_b, -x_0 \gamma_b, y_0, 0)$$

$$e_3' = e_3 B_- = (0, -x_0, -y_0, 0) \begin{bmatrix}
\gamma_b & -v_b \gamma_b & 0 & 0 \\
-v_b \gamma_b & \gamma_b & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = (x_0 v_b \gamma_b, -x_0 \gamma_b, -y_0, 0)$$

$$e_4' = e_4 B_- = (0, x_0, -y_0, 0) \begin{bmatrix}
\gamma_b & -v_b \gamma_b & 0 & 0 \\
-v_b \gamma_b & \gamma_b & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} = (-x_0 v_b \gamma_b, x_0 \gamma_b, -y_0, 0),$$

where we still identify $e_1'$ with $e_2'$ and $e_3'$ with $e_4'$.

We will now produce a CTC by letting a space machine travel with velocity $v_s < 1$ (speed of light set to $c = 1$) between these events in a closed circular manner as: $e_1' \rightarrow e_2' \rightarrow e_3' \rightarrow e_4' \rightarrow e_1'$. Starting at $e_1'$, we are by identification also at $e_2'$. The separation between $e_2'$ and $e_3'$ is spatially $\Delta y = 2y_0$, and in the time-coordinate $t$ is $\Delta t = 2x_0 v_b \gamma_b$. For the space machine with given velocity $v_s$ to be able to travel this, we need that the path distance equals the speed multiplied with the time interval, $v_s \Delta t = \Delta y$,

$$2 x_0 v_s v_b \gamma_b = 2 y_0,$$

which with (3.64) reads, by setting the explicit form of $\theta = 4 \pi \mu$,

$$v_s v_b \gamma_b \cos(4 \pi \mu) = c \cos(4 \pi \mu) + d,$$

$$v_s v_b \gamma_b = \frac{d}{c \sin(4 \pi \mu)} + \frac{\cos(4 \pi \mu)}{\sin(4 \pi \mu)}$$

(3.66)

as we stated earlier directly after equation (3.62), $c > d$ must hold for the timelike travel to be possible. But then we might as well choose $d \ll c$, so that the first term in (3.66) vanishes, leaving us

$$v_s v_b \gamma_b = \frac{\cos(4 \pi \mu)}{\sin(4 \pi \mu)},$$

(3.67)

but we had that the space machine is a massive object traveling on a timelike trajectory, so $v_s < 1$, yielding

$$v_b \gamma_b > \frac{\cos(4 \pi \mu)}{\sin(4 \pi \mu)},$$

(3.68)
but we are also making a boost with a not superluminal speed, so \( v_b < 1 \), giving

\[
\gamma_b > \frac{\cos(4\pi\mu)}{\sin(4\pi\mu)}.
\]

(3.69)

and we also know \( \cos(4\pi\mu) \in [0, 1] \), so certainly

\[
\gamma_b > \frac{1}{\sin(4\pi\mu)},
\]

(3.70)

which certainly is possible, since \( \sin(4\pi\mu) \in [0, 1] \), and so \( \gamma_b \geq 1 \), just as it should be.

From \( e'_3 \), we are identically also at \( e'_4 \) due to the identification of the points after the boost. From \( e'_4 \) we can travel back to \( e'_1 \) in the same manner, as the time and spatial separation are the same as between \( e'_2 \) and \( e'_3 \). So indeed, we traveled along a timelike trajectory of the space machine, and ended at the initial event \( e'_1 \), which gives us a closed timelike curve.

\subsection*{3.3.2 Possible problems}

The cosmic string model of producing CTCs is considered to contain some flaws both in the mathematics and in the question of it being a physically possible situation.

Firstly, notice that we claimed the events \( e_i \) to be identified even after the Lorentz transformation because their spatial coordinates satisfy the identification criterion. However we note also that their temporal coordinate does not match. In a sense, they will then live on two different Euclidean spaces, at two different slices of time, and whether they can then still be identified is questionable, as we have stated nothing about the relation of the time coordinates in the identification.

Another question we can ask is whether we could ever reach the level at which the final expression (3.70) holds. According to research done by Gott, Hawking and Thorne about the formation of black holes [11], the two by-passing cosmic strings at such high velocities might actually form a single black hole before the CTC can appear by pulling the masses of the two strings together to a black hole mass.

A third question arising is whether this model of infinite cosmic strings is realistic. In physical systems, infinity does not seem to exist so far in our detections, so an infinite string is somewhat unphysical. The closest in reality we have is the case of a finite string. There is however nothing guaranteeing that an infinite model is only infinitesimally related to the finite, realistic model. Realistic, finite strings therefore might in real physical spaces not satisfy the same equations as are derived for the metric of infinite cosmic strings. The solution for a finite string can be equated in a similar manner, however is more difficult to perform analytically and thus is omitted in this report. Calculating numerically the metric for the finite case might reveal however if CTCs truly exist in that spacetime as well.

\subsection*{3.4 Paradoxes of CTCs}

Why is the physics world trying to avoid closed timelike curves in the solutions of EFEs? CTCs essentially provide a possibility for time travel to the past. For example, an observer enters a closed timelike curve at his birth and stays in it during his entire life, and then at some point the person would suddenly realize he has arrived back to the moment of his birth, the point where the CTC closes into
its initial point. This way, the observer travels back in time. This possibility gives rise to some questions.

Firstly, consider that the entire world goes into a CTC at an early stage of one’s ancestors life. The world then develops as usual until the person in question is born. Then suddenly the CTC closes into its initial point by returning to the past moment of the early stage of the person’s ancestors life. The person can, assuming that he is free to undergo any action, cause death of his ancestor. Will the person cease to exist directly or not at all? This is the **grandfather paradox** and evokes questions about being able to affect the past in ways which will cause the present (or future) to change.

Another paradox arises when we assume there is an absolute free will for the traveler on a CTC. Suppose an observer A meets an older version of himself, B. B has been traveling on a CTC and thus has been aging, but could get back to the initial point of his departure. B tells A how to travel on the CTC, allowing thus for A to depart on the CTC. Now when A arrives to the same point as B was earlier, he meets the older version of himself A (in a sense, A becomes B along the CTC). However this time, assuming he can do so, decides not to share the information about the secret of the CTC. Therefore, A will not be able to travel on the CTC. But then who will have told the initial A how to travel on the CTC in first place? Did the CTC then never even exist? We will refer to this paradox as the **free will paradox**.
4 Quantum mechanical closed timelike curves

Finally, we discuss the paradoxes of CTCs in a quantum mechanical manner. We can always decompose a complex particle into many small carriers of information. For example, if we have a green circular soft particle then we could decompose it into three carriers, where each carries information about one of the properties. In a 2-bit system, we would have the first carrier in state 1 for saying that it is green rather than the red 0; the next carrier in state 0 because particle is circular and not rectangular for 1; and lastly a carrier of 1 for soft instead of hard 0. This way we can decompose a system into subcarriers which can be in two different states.

Next, the notion of interaction among particles is symbolized by a gate in the network: this can be any interaction such as the particles colliding, the particles sharing information or just looking at each other. A gate will be denoted with a pink G in the network figures, with states going into it and states coming out.

We will assume some forms of the interactions in the networks. A feature being used is the "+"-sign. This is a module-summation, meaning that, working in a 2-state basis of 0 and 1 possible values, we add them as: 0+0=0, 0+1=1, 1+1=0, so the result is the usual answer modulus 2.

The CTC in the networks is denoted with a blue box labeled ",-T", referring to a backwards time travel in a sense that the one travels back to the initial coordinate time.

4.1 Classical networks

Firstly, how would we portray time travel along CTCs with networks? Assume for this first case that the states to be encountered are in the basis of the 2-state space. Thus, this corresponds to a classical system, not allowing any superpositioned states, following the procedure of [15].

We will firstly consider some examples of specific interactions. As a first example, a subcarrier which is to travel on a CTC enters a gate G in a state \( |\alpha \rangle \) where it interacts with itself after having returned on a CTC, being in a state \( |\beta \rangle \). The original carrier emerges after the interaction as state \( |\alpha + \beta \rangle \), denoting some kind of interaction, and the already time traveled state merges as state \( |\beta \rangle \), without any change, see Fig. 9 for the network. The following interaction takes place, with the first ket state for the first particle, and the second ket state for the second particle,

\[
|\alpha \rangle |\beta \rangle \rightarrow |\alpha + \beta \rangle |\beta \rangle .
\] (4.1)

However no interaction occurs, by assumption, outside G, so the state passing out and coming in along the CTC has to be the same,

\[
|\beta \rangle = |\alpha + \beta \rangle .
\] (4.2)

If the states are to be in the basis of the 2-state system, they are either 0 or 1. However, if \( \alpha = 1 \), then no \( \beta \) can solve this problem, given the definition of the module-summation. Therefore, this interaction takes place only if \( \alpha = 0 \), in which case both values of \( \beta \) solves it.

To be able to obtain the consistency of the situation, we received that only some initial values of the incoming carrier is allowed. This provides a possible resolution of the CTC paradoxes in the classical
Figure 9: Classical networks for two different examples of interactions: interaction (4.1) (left) and interaction (4.3) (right). Classically, only the first of these interactions is possible for a given initial value $\alpha = 0$.

regime: we no longer have the complete free will to undergo the CTCs, but only if we are in some state can we travel in time. This is a consistency condition imposed, and looking at the grandfather paradox, this would translate into that only those time travelers which are not in intention of causing death of their ancestors can travel on a CTC.

Now let us consider another interaction, portrayed in Fig. 9, given by

$$ |\alpha\rangle |\beta\rangle \rightarrow |\beta\rangle |\beta\rangle .$$

(4.3)

Now the condition of nothing changing outside the interaction is:

$$ |\beta\rangle = |\beta\rangle ,$$

(4.4)

which is not satisfied by any $\beta$.

Such a network classically represents a time travel being made in the following way: the particle just entering a CTC becomes smarter in a sense than it is by the time it gets to the interaction with its old self. This way, the old self interacts with a less informed time traveler, than the one which emerges from the interaction. The time traveler on purpose forgets to deliver some information. This is classically as we see not permitted at the level of the subcarriers, since we can have no state for which the consistency condition (4.4) holds. This is the free will paradox which we discussed earlier, and we see that this cannot take place in a classical regime, when analyzing it with a 2-state system of subcarriers.

Let us now develop a different classical network, where the 1 state represents the presence of a particle in that state, and the 0 denotes the lack of the particle. We have then two possible going trajectories into the gate: one which will continue on a CTC, and one which will not. Schematically the network is shown in Fig. 10, for the specific interaction to be presented. The two incoming paths are with states $|\alpha\rangle$ representing a state which does not know about CTCs, and a more informed state $|\alpha + 1\rangle$, which can potentially travel on the CTC. The state which has already come back from its CTC-travel is in state $|\beta\rangle$. The uninformed state interacts with the CTC-traveler, as well as the informed one. This is an example of an interaction given by

$$ |\alpha\rangle |\alpha + 1\rangle |\beta\rangle \rightarrow |\alpha + \beta\rangle |\alpha + \beta + 1\rangle |\beta\rangle ,$$

(4.5)

and the consistency condition gives

$$ |\beta\rangle = |\alpha + \beta + 1\rangle .$$

(4.6)
If $\alpha = 1$, then this reduces to the equation (4.4), which has no solutions for $\beta$. Thus we can only have $\alpha = 0$, in which case both $\beta = 0$ and $\beta = 1$ solve the equation. The two cases are shown in Fig. 11. In the case $\beta = 0$, there is no time travel being made, and the particle passes the non-CTC trajectory without any interaction with any CTC-traveler carrier. For the case $\beta = 1$, the carrier passing on the trajectory with no intention of traveling on CTC, interacts with the older version of the CTC-traveler, and by influence instead changes path to the CTC. These two cases are fully possible classically.

**Figure 10:** Classical network with interaction given by (4.5). Now the states 1 and 0 represent the existence or the non-existence of the carrier on the given trajectory.

**Figure 11:** Classical network showing two solutions to the consistency condition (4.6) with the necessary condition $\alpha = 1$. Red marking shows the path of the particle for the two cases $\beta = 0$ (left) and $\beta = 1$ (right).

### 4.2 Quantum mechanical networks

Let us now consider the states with quantum mechanical approach, no longer demanding the states to be in one of the basis of the 2-state system, but in any arbitrary superposition state of these

$$|s\rangle = a_0 |0\rangle + a_1 |1\rangle.$$  

We wish to prove that quantum mechanics does not set boundaries for the initial condition for a CTC as in the classical case of the two interactions and for any arbitrary unitary interaction.

Now we will consider the density operator of the states, regarding them as a quantum mechanical ensemble with different states in the system.
Also, we need to describe the interaction as a quantum mechanical operator. By assumption, let it be an operator conserving the norm of the states, so a unitary operator $U$, such that $UU^\dagger = 1$. The operator acts on a density operator $\rho$ as: $U \rho U^\dagger$.

Let us consider interaction (4.1) first. We want the unitary operator $U_1$ to give exactly the transformation (4.1) when acting on the basis vectors. The one satisfying this is

$$U_1 = \sum_{\alpha'} \sum_{\beta'} |\alpha' + \beta'| \bra{\alpha'} \bra{\beta'} |\beta'\rangle \langle \alpha' |$$  \hspace{1cm} (4.8)

where the sum over the dummy indices goes over all possible states of the bases. We see that this satisfies the interaction by acting this on the incoming direct product state

$$U_1 |\alpha \rangle |\beta\rangle = \sum_{\alpha'} \sum_{\beta'} |\alpha' + \beta'| \bra{\alpha'} \bra{\beta'} |\beta'\rangle \langle \alpha' | \langle \alpha | \beta\rangle = \langle \alpha + \beta | \beta\rangle ,$$  \hspace{1cm} (4.9)

just as the interaction was defined on the bases. We can also explicitly check that the Hermitian conjugate of this unitary operator is

$$U_1^\dagger = \sum_{\alpha'} \sum_{\beta'} |\alpha' \rangle \langle \beta' \rangle |\alpha' + \beta'| \langle \beta' | .$$  \hspace{1cm} (4.10)

Let the incoming state be a general $|s\rangle$, with a density operator $\rho_s = |s\rangle \langle s|$. The state of the carrier coming from the CTC is $|q\rangle$ with density operator $\rho_q = |q\rangle \langle q|$. The total density operator is $\rho = \rho_s \otimes \rho_q$. The gate $G$ acts on the total density operator, giving the outgoing density operator $\rho'$ as

$$\rho' = U_1 (\rho_s \otimes \rho_q) U_1^\dagger .$$  \hspace{1cm} (4.11)

The consistency condition we discussed in the classical networks translates into that the second factor in the direct product of the final density operator $\rho'$ be equal to the incoming CTC-traveled density operator $\rho_q$. Mathematically this corresponds to taking the partial trace of the final density operator over the subspace of the outgoing state (of the CTC carrier), (see in more mathematical detail in Appendix C). We use the index $q$ of the partial trace over this subspace to denote that we are tracing over all outgoing versions of the state $|q\rangle$,

$$Tr_q[U_1(\rho_s \otimes \rho_q) U_1^\dagger] = \rho_q .$$  \hspace{1cm} (4.12)

By replacing the exact form of the unitary transformation (4.8), and assuming a given ingoing state $|s\rangle$, this equation is solved by

$$\rho_q = \frac{1}{2} \mathbb{1} + \text{Re}[\langle 0| s \rangle \langle s | 1\rangle] (|0\rangle \langle 0| + |1\rangle \langle 1|).$$  \hspace{1cm} (4.13)

See Appendix C for verification of this being a solution of (4.12).

We see now however that this sets no constraint on the initial state $|s\rangle$: this can even be the classically forbidden state $|s\rangle = |1\rangle$, since then the consistency condition (4.12) still holds for $\rho_q = \frac{1}{2} \mathbb{1}$. 38
Next, let us consider the second interaction (4.3), for which case the unitary operator and its Hermitian conjugate is

\[
U_2 = \sum_{\alpha'} \sum_{\beta'} |\beta' + 1 \rangle |\beta' \rangle \langle \alpha'| \langle \beta'|,
\]
\[
U_2^\dagger = \sum_{\alpha'} \sum_{\beta'} |\alpha' \rangle |\beta' \rangle \langle \beta' + 1 \rangle \langle \beta'|,
\]

(4.14)

which, with the same notations for the incoming states \(\rho_s\) and \(\rho_q\), leads to the same consistency condition as with the first interaction, only the unitary operator changed,

\[
Tr_q[U_2(\rho_s \otimes \rho_q)U_2^\dagger] = \rho_q.
\]

(4.15)

This equation has solution

\[
\rho_q = \frac{1}{2} \mathbb{1} + \frac{1}{2} m (|0\rangle \langle 1| + |1\rangle \langle 0|), \quad \text{with} \quad m \in [0, 1]
\]

(4.16)

which can be verified in the same manner as the verification of (4.13) given in Appendix C.

Finally we turn to the most general case of CTCs in quantum networks, with the interpretation of 0 being the lack of a particle and 1 being the existence of a particle, as in the classical consideration of interaction (4.5). Let there be an arbitrary number of carriers for both the CTC-traveling trajectory and the not CTC-traveling trajectory, with each being considered as a pure quantum ensemble with the density operator \(\rho_{no}\) (no CTC-travel) respective \(\rho_{CTC}\), see Fig. 12. The total ensemble density operator is \(\rho_{no} \otimes \rho_{CTC}\).

\[\text{Figure 12: Quantum network with arbitrary ingoing particles, in a quantum ensemble with density operator } \rho_{no} \text{ which do not travel on a CTC, and an ensemble of particles indeed traveling on a CTC with density operator } \rho_{CTC}.\]

The gate is an arbitrary unitary operator \(U\), with Hermitian conjugate \(U^\dagger\). The total outgoing state from the interaction is \(\rho'\) given by

\[
\rho' = U(\rho_{no} \otimes \rho_{CTC})U^\dagger.
\]

(4.17)

The consistency condition for this general case is received by taking the partial trace of the \(\rho'\) over the subspace of the outgoing ensemble of the not CTC-traveling particles, over the space which we will denote with the index \(no\) (for no CTC-travel),

\[
Tr_{no} (U(\rho_{no} \otimes \rho_{CTC})U^\dagger) = \rho_{CTC}.
\]

(4.18)
Our aim now is to prove that this equation has a solution for all states \( \rho_{\text{no}} \), so that there is no constraint on the initial condition.

Following the procedure of Deutsch [15], we define a scattering operator \( S \), acting on a density operator \( P \) as

\[
S(P) := \text{Tr}_{\text{no}} \left( U (\rho_{\text{no}} \otimes P) U^\dagger \right),
\]

which means that we are really looking for a fixed point of this operator,

\[
S(P) = P.
\]

We build a series of densities \( \{ \rho''_N \} \) with elements consisting of the operator \( S \) acting on an initial density operator \( \rho_0 \) increasing number of times for each element, and then summing over them,

\[
\rho''_N = \sum_{i=0}^{N} S^i \rho_0,
\]

and yet another sum \( \rho_N \) which we average over the number of actions,

\[
\rho_N = \frac{1}{N+1} \sum_{i=0}^{N} S^i \rho_0.
\]

This choice of sequence is done just in order to complete the proof to follow, but we can also give it a physical interpretation. The elements \( \rho_N \) are simply the average of networks with \( N \) scatterings, that is, with \( N \) CTCs, see Fig. 13.

**Figure 13:** Schematic figure of the interpretation of the \( \rho_N \) sequence elements. For each \( N \), we obtain \( \rho_N \) by summing over all the scatterings with \( N \) number of CTCs, normalized. In the figure we plot the networks for the first few \( N \)s.

At this point we define an average value sequence \( E(\rho_N) \), as the difference between the average and the one-scattering of the average. As the scattering operator acting on a density operator yields a density
operator, we need to take the trace of this density operator, in order to have a sequence with numbers rather than operators,

\[ E(\rho_N) := \text{Tr}[(S\rho_N - \rho_N)^2], \quad (4.23) \]

and our aim (4.20) translates then to finding a point \( P = \rho_N \), such that \( E(\rho_N) = 0 \). We can simplify the average \( E \) by using the definition of the \( \rho_N \) sequence, (4.22),

\[
E(\rho_N) = \text{Tr} \left[ \left( \frac{1}{N+1} \sum_{i=0}^{N} S^{i+1} \rho_0 - \frac{1}{N+1} \sum_{i=0}^{N} S^i \rho_0 \right)^2 \right] = \\
= \left( \frac{1}{N+1} \right)^2 \text{Tr}[(S^{N+1}\rho_0 - \rho_0)^2]. \quad (4.24)
\]

The operator \( S \) reduces always the operator it acts on, as it traces out part of the outgoing states. Thus, we have an upper bound on the trace

\[ \text{Tr} [(S^{N+1}\rho_0 - \rho_0)^2] \leq 1, \quad (4.25) \]

upon which we find, together with the non-negativity of the definition (4.23)

\[ 0 \leq E(\rho_N) \leq \left( \frac{1}{N+1} \right)^2, \quad (4.26) \]

which means that our sequence \( \{E(\rho_N)\} \) is bounded with upper and lower bound. This space on which we define our sequence is both closed and bounded, meaning it is compact topologically. But all sequences on compact spaces include their limit points. The sequence \( \{E(\rho_N)\} \) however converges to 0 as \( N \to \infty \), and thus it must contain its limit point \( \rho_C \) at least once in the sequence. For all those elements, \( E(\rho_C) = 0 \) and thus solves \( S(\rho_C) = \rho_C \) and thus the equation (4.18). This means that we can always find such a density operator which satisfies (4.18), for any \( \rho_{\text{no}} \), since we place no constraint on this ingoing density operator. The implication of this is that quantum mechanics sets no constraint on the initial conditions \( \rho_{\text{no}} \) for any arbitrary unitary interaction \( U \). This is opposed to the classical case, for as we saw the example interaction (4.5), we had classically a clear constraint \( \alpha = 0 \) for the CTC to occur. With quantum mechanics, this constraint is ruled out and CTCs can occur for any ingoing particle!

5 Discussion

We have seen that CTCs arise in different spacetimes when considering them with classical general relativity. However, in all solutions there are some problems either with the mathematical correctness, or with the physical reality measure of it. Quantum mechanically however we see that CTCs place no constraints on initial conditions, and they are indeed possible objects. In many areas of physics, researchers are impelled to undertake quantum mechanical approaches to be able to understand reality, when classical physics does not suffice. The double slit experiment, the photoelectric effect are examples of such occasions. Perhaps the subject of CTCs craves, just as these topics do, the deeper analysis of quantum mechanics to (almost) fully explain reality with physics. There is really nothing that entitles us to think that classical physics is the correct description of CTCs, and noting all the paradoxes classical physics leads to in CTCs, we are rather triggered to believe that quantum mechanics should be considered.
However in this subject we have no reality to compare physics to. Would we for sure know CTCs exist, then there would be not as much debate about whether quantum mechanics should be considered or not.

In derivations of the CTCs in the classical spacetimes of the rotating black hole and the Gödel universe we claimed that identifying the angular coordinate $\phi$ when it becomes timelike leads to problems such as the existence of CTCs in flat spacetime. This is however only a problem if we claim that wherever CTCs arise, they should be detected. There is a long debate in the physicists’ society whether the coordinate behaviors can be adapted once they change causal structure. Mathematically there is no boundary on this: it is fully legitimate to impose identification of the angular coordinates inside the Kerr black hole, and in the Gödel universe, up to level of mathematical strictness on which this report is based on. A future work is to study the behavior of coordinate transformations from a strict mathematical perspective and see if this really provides any problem mathematically on a deeper level, or if this question of identification is merely a physical question of chronology-protecting assumptions.

We can still ask ourselves whether we can truly detect CTCs if they exist at any point. Can there be some mechanism forbidding observers to detect CTCs? Such mechanism is proposed in the Gott’s cosmic string solution, where we mentioned that the strings passing each other might collapse to form a black hole instead. This would hide the CTC behind an event horizon and perhaps even cease it to exist before it can be detected and used. Other mechanisms are also plausible, as the energy needed to maintain a CTC (for some reason) extends the total energy in the universe. At the present time there is no known mechanism which would forbid backward time travel, but many physicists claim that this could and should be the case. We can similarly assume as a conclusion of this project, that such mechanisms could exist, leaving all other aspects of CTCs to conspire. Mathematically thus, all derivations are correct, and CTCs actually do exist in our space of (nearly) flat spacetime, but this mechanism forbids us to detect it. However this mechanism is not needed for the quantum mechanical approach of it, instead there is a mechanism not letting us detect purely quantum mechanical systems, making the gap between observations and quantum mechanical CTCs impossible to tread over.

Another interesting spacetime is the anti-de Sitter space, where CTCs arise. In this spacetime one can study the behavior of the global coordinates, which provide a universal covering of the spacetime and which provide the disappearance of CTCs. The question we might ask is whether this mathematical coordinate change really rules out CTCs or it is just a wit of mathematics. But to be able to answer this, the spacetime needs to be studied, which this project did not extend to, but warmly suggests a future work in this topic.

Much future work can be dedicated to the field of CTCs. Firstly, no work on the numerical solutions of EFEs and CTCs was covered in this report. Numerical solutions are in many cases more accurate physically if one solves complex systems in more proximity of the observable world, giving us an insight to more probable occurrences of CTCs. Secondly, there is a possibility for more work in the investigation of the Gott’s cosmic string production of CTCs, where we left the question of identifying events after the boost of the string open for discussion. Lastly, as the black hole creation as a possible mechanism for the forbidding of CTC-detection is one of the best options seen in this project, a thorough investigation on how such black hole formation can come about and how it is in relation with the usual information paradox and the cosmic censorship conjecture, can perhaps lead to better understanding of the nature of CTCs, which we hereby claim are existing but are unable to be observed.
Appendices

A Derivation of Einstein’s field equations

In this section we derive Einstein’s field equations using the least action principle.

We wish to find equations of motion for the metric. This is received by demanding the variation of the action with respect to small variations in the metric to disappear. We take the variation of $S$ with respect to $g^{\mu\nu}$, using the chain rule. We first consider the Hilbert-action variation:

$$\delta S_H = \int d^4x (\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} + \sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu} + \delta \sqrt{-g}g^{\mu\nu}R_{\mu\nu}) \stackrel{!}{=} 0 \quad (A.1)$$

Variation of Riemann tensor is:

$$\delta R_{\mu\lambda\nu} = \partial_{\lambda}\delta \Gamma_{\nu\mu}^{\rho} - \partial_{\nu}\Gamma_{\mu\lambda}^{\rho} + \delta(\Gamma_{\lambda\rho}^{\nu}\Gamma^{\mu}_{\nu}) - \delta(\Gamma_{\nu\rho}^{\mu}\Gamma_{\lambda}^{\nu}) =$$

$$= \partial_{\lambda}\delta \Gamma_{\nu\mu}^{\rho} - \partial_{\nu}\Gamma_{\mu\lambda}^{\rho} + \Gamma_{\rho\lambda}^{\sigma}\delta \Gamma^{\rho}_{\sigma\mu} + \Gamma_{\rho\sigma}^{\mu}\delta \Gamma^{\rho}_{\nu\lambda} - \Gamma_{\rho\nu}^{\mu}\delta \Gamma^{\rho}_{\lambda\mu} - \Gamma_{\rho\mu}^{\nu}\delta \Gamma^{\rho}_{\lambda\nu} \quad (A.2)$$

Now the covariant derivatives of the variation of the Christoffel symbols are, for two different index placements:

$$\nabla_{\nu}\delta \Gamma^{\rho}_{\mu\lambda} = \partial_{\nu}\delta \Gamma^{\rho}_{\mu\lambda} + \Gamma_{\nu\sigma}^{\rho}\delta \Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\sigma}_{\mu\sigma}\delta \Gamma^{\rho}_{\nu\lambda}$$

$$\nabla_{\lambda}\delta \Gamma^{\rho}_{\nu\mu} = \partial_{\lambda}\delta \Gamma^{\rho}_{\nu\mu} + \Gamma_{\lambda\sigma}^{\rho}\delta \Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\nu\sigma}\delta \Gamma^{\rho}_{\lambda\mu}$$

We recognize the difference between these two covariant derivatives as the variation of Riemann tensor as given in expression (A.2):

$$\nabla_{\lambda}\delta \Gamma^{\rho}_{\nu\mu} - \nabla_{\nu}\delta \Gamma^{\rho}_{\mu\lambda} = \partial_{\lambda}\delta \Gamma^{\rho}_{\nu\mu} + \Gamma_{\lambda\sigma}^{\rho}\delta \Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\nu\sigma}\delta \Gamma^{\rho}_{\lambda\mu} - (\partial_{\nu}\delta \Gamma^{\rho}_{\mu\lambda} + \Gamma_{\nu\sigma}^{\rho}\delta \Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\sigma}_{\mu\sigma}\delta \Gamma^{\rho}_{\nu\lambda}) =$$

$$= \delta R_{\mu\lambda\nu} \quad (A.3)$$

So this will lead to a variation of the Ricci tensor:

$$\delta R_{\mu\nu} = \delta R_{\mu\rho\nu} = \nabla_{\nu}\delta \Gamma^{\rho}_{\mu\rho} - \nabla_{\mu}\delta \Gamma^{\rho}_{\rho\nu}$$

and placing in $\delta S_1$ in equation (A.1):

$$\delta S_1 = \int d^n x \sqrt{-g}g^{\mu\nu}(\nabla_{\nu}(\delta \Gamma^{\rho}_{\mu\rho}) - \nabla_{\mu}(\delta \Gamma^{\rho}_{\rho\nu})) =$$

$$= \int d^n x \sqrt{-g}(g^{\mu\nu}\nabla_{\nu}(\delta \Gamma^{\rho}_{\mu\rho}) - g^{\mu\nu}\nabla_{\nu}(\delta \Gamma^{\rho}_{\rho\nu})) =$$

$$= \int d^n x \sqrt{-g}\nabla_{\sigma}(g^{\mu\nu}(\delta \Gamma^{\sigma}_{\mu\nu}) - g^{\mu\sigma}\nabla_{\sigma}(\delta \Gamma^{\rho}_{\rho\nu})) =$$

$$= \int d^n x \sqrt{-g}\nabla_{\sigma}(g^{\mu\nu}(\delta \Gamma^{\sigma}_{\mu\nu}) - g^{\mu\sigma}(\delta \Gamma^{\rho}_{\rho\nu})) \quad (A.3)$$
Taking the variation of the Christoffel symbols with respect to the metric, recalling the explicit form of the torsion-free Christoffel symbol (2.28):

\[
\delta \Gamma^\rho_{\mu \nu} = \frac{1}{2} \left[ \delta (g^\sigma \nabla_\rho g_{\nu \lambda}) + \delta (g^\sigma \nabla_{\nu} g_{\lambda \rho}) - \delta (g^\sigma \nabla_\lambda g_{\mu \nu}) \right] = \\
= \frac{1}{2} \left[ (\delta g^\sigma \nabla_\mu g_{\nu \lambda} + g^\sigma \nabla_\mu \delta (g_{\nu \lambda}) + (\delta g^\sigma \nabla_{\nu} g_{\lambda \rho}) + g^\sigma \nabla_{\nu} (\delta g_{\lambda \rho} - \delta g_{\lambda \rho} - \delta g^\lambda \nabla_\lambda (\delta g_{\mu \nu})) \right] = \\
= \frac{1}{2} [g^\sigma \nabla_\mu \delta (g_{\nu \lambda}) + g^\sigma \nabla_\nu (\delta g_{\lambda \rho}) - g^\sigma \nabla_\lambda (\delta g_{\mu \nu})] \\
\text{(A.4)}
\]

where we have invoked metric compatibility of the covariant derivative.

Variation of the inverse metric can be computed, starting with the definition of the inverse metric:

\[
g^{\mu \nu} g_{\nu \lambda} = \delta^\mu_\lambda \\
\Rightarrow \delta (g^{\mu \nu} g_{\nu \lambda}) = 0 \\
\Rightarrow (\delta g^{\mu \nu}) g_{\nu \lambda} + g^{\mu \nu} (\delta g_{\nu \lambda}) = 0 \\
\Rightarrow (\delta g^{\mu \nu}) g_{\nu \lambda} = -g^{\mu \nu} (\delta g_{\nu \lambda}) \\
\Rightarrow g_{\mu \sigma} (\delta g^{\mu \nu}) g_{\nu \lambda} = -g_{\mu \sigma} g^{\mu \nu} (\delta g_{\nu \lambda}) \\
\Rightarrow g_{\mu \sigma} (\delta g^{\mu \nu}) g_{\nu \lambda} = -\delta^\sigma_\nu (\delta g_{\nu \lambda}) \\
\Rightarrow \delta g_{\mu \nu} = -g_{\sigma \mu} g_{\lambda \nu} \delta g^{\mu \nu} \text{ (A.5)}
\]

which conventional tensor manipulation suggests as well. Plugging this in variation of Christoffel symbol (A.4):

\[
\delta \Gamma^\sigma_{\mu \nu} = \frac{1}{2} \left[ g^\sigma \nabla_\mu (g_{\nu \alpha} g_{\lambda \beta} \delta g^{\alpha \beta}) + g^\sigma \nabla_\nu (-g_{\mu \rho} g_{\mu \tau} \delta g^{\rho \tau}) - g^\sigma \nabla_\lambda (-g_{\mu \tau} g_{\nu \omega} \delta g^{\tau \omega}) \right] = \\
= \frac{1}{2} [\delta g^\sigma g_{\nu \alpha} g_{\lambda \beta} (\nabla_\mu \delta g^{\alpha \beta}) - g^\sigma \nabla_\nu g_{\mu \rho} (\nabla_\tau \delta g^{\rho \tau}) + (\delta g^\sigma \nabla_\lambda g_{\mu \tau}) (\nabla \delta g^{\tau \omega})] = \\
= \frac{1}{2} [\delta g^\sigma g_{\nu \alpha} (\nabla_\mu \delta g^{\alpha \beta}) - \delta^\rho_\mu (g_{\nu \tau} \delta g^{\rho \tau}) + (\delta g^\sigma \nabla_\lambda g_{\mu \tau}) (\nabla \delta g^{\tau \omega})] = \\
= \frac{1}{2} [g^\sigma \nabla_\mu (\delta g^{\alpha \beta}) - g_{\mu \tau} (\nabla_\nu \delta g^{\rho \tau}) + g_{\mu \tau} g_{\nu \omega} \nabla \delta g^{\tau \omega}] \\
\text{(A.6)}
\]

Plugging this back into the first variation \(\delta S_1\) (A.3):

\[
\delta S_1 = \int d^n x \frac{1}{2} \sqrt{-g} \nabla_\sigma [g^{\mu \nu} ((-g_{\nu \alpha} (\nabla_\mu \delta g^{\alpha \sigma}) - g_{\mu \tau} (\nabla_\nu \delta g^{\tau \sigma}) + g_{\mu \tau} g_{\nu \omega} \nabla \delta g^{\tau \omega})) - \\
- g^{\mu \sigma} ((-g_{\lambda \rho} (\nabla_\mu \delta g^{\rho \lambda}) - g_{\mu \tau} (\nabla_\lambda \delta g^{\tau \rho}) + g_{\mu \tau} g_{\lambda \omega} \nabla \delta g^{\tau \omega})) = \\
= \int d^n x \frac{1}{2} \sqrt{-g} \nabla_\sigma [\left(-\delta^\sigma_\alpha \nabla_\mu \delta g^{\alpha \sigma} \right) - \delta^\rho_\mu (\nabla_\nu \delta g^{\rho \tau}) + (\delta g^\sigma \nabla_\lambda g_{\nu \omega} \nabla \delta g^{\tau \omega})] + \\
+ [g^{\mu \sigma} g_{\lambda \rho} (\nabla_\mu \delta g^{\rho \lambda}) + \delta^\epsilon_\nu (\nabla_\lambda \delta g^{\epsilon \tau}) - \delta^\sigma_\nu g_{\lambda \omega} \nabla \delta g^{\tau \omega}] = \\
= \int d^n x \frac{1}{2} \sqrt{-g} \nabla_\sigma [\left(-\delta^\sigma_\alpha \nabla_\mu \delta g^{\alpha \sigma} \right) - \delta^\rho_\mu (\nabla_\nu \delta g^{\rho \tau}) + g_{\nu \omega} \nabla \delta g^{\tau \omega})] + \\
+ g_{\lambda \rho} (\nabla \delta g^{\alpha \lambda}) + (\nabla_{\mu} \delta g^{\rho \tau}) - \nabla_{\omega} \delta g^{\tau \omega}] = \\
= \int d^n x \frac{1}{2} \sqrt{-g} \nabla_\sigma [\left(-2\nabla_\mu \delta g^{\alpha \sigma} + 2g_{\nu \omega} \nabla \delta g^{\tau \omega}\right)] \\
\text{(A.6)}
\]
For this we use Stokes’ theorem, which in a general form is:

\[ \int_{\Sigma} \nabla \mu V^\mu \sqrt{|g|} d^n x = \int_{\partial \Sigma} n_\mu V^\mu \sqrt{|\gamma|} d^{n-1} x \quad (A.7) \]

where \( \gamma \) is induced metric’s determinant on the boundary surface \( \delta \Sigma \), and \( n_\mu \) is the normal vector on this boundary surface and the covariant derivative \( \nabla \mu \) is the derivative over the volume element components. We can apply Stokes’ theorem to our expression \((A.6)\), with an integrand vector field:

\[ V^\sigma = -2 \nabla \mu \delta g^\mu \sigma + 2 g_{\nu \omega} \nabla^\sigma \delta g^{\nu \omega} \]

Evaluated at the boundary of spacetime, we can set the variation of the metric to be zero where we assume an asymptotic flatness to take over. According to Stokes’ theorem then, the entire \( \partial S_1 \) will vanish in the Hilbert action.

Now turn to last part in Hilbert action \((A.1)\), rewriting it:

\[ \partial S_3 = \int d^n x \partial \sqrt{-g} = \int d^n x R(-\frac{1}{2} (-g)^{-1/2} \delta g) \]

To calculate \( \partial g \), we make use of the following definition of the determinant \(|A|\) of a square matrix \( A \), given in [1]:

\[ \log(|A|) = Tr(\log(A)) \quad (A.8) \]

In the case of the metric \( g_{\mu \nu} \), being a square matrix, denoting \(|g| := g\):

\[
\begin{align*}
\log(g) &= T r(\log(g_{\mu \nu})) \\
\Rightarrow \delta(\log(g)) &= \delta(T r(\log(g_{\mu \nu}))) \\
\Rightarrow \frac{1}{g} \delta g &= T r(\frac{1}{g_{\mu \nu}} \delta g_{\mu \nu}) \\
\Rightarrow \delta g &= g T r(g^{\sigma \rho} \delta g_{\sigma \rho}) = g(g^{\mu \nu} \delta g_{\mu \nu})
\end{align*}
\]

(A.9)

Invoke expression \((A.5)\) for the variation of the metric:

\[
\delta g = g(g^{\mu \nu} (-g_{\mu \alpha} g_{\nu \beta} \delta g^{\alpha \beta})) =
\]

\[= -g(g^{\mu \nu} \delta g^{\alpha \beta}) =
\]

\[= -g(g_{\alpha \beta} \delta g^{\alpha \beta}) \]

(A.10)

Replacing this expression \((A.10)\) into \((A.1)\):

\[
\partial S_3 = \int d^n x R(-\frac{1}{2} (-g)^{-1/2} (-g(g_{\alpha \beta} \delta g^{\alpha \beta}))) =
\]

\[= \int d^n x R(-\frac{1}{2} \sqrt{-g} g_{\alpha \beta} \delta g^{\alpha \beta}) \]

(A.11)

Now putting all nonvanishing parts of the Hilbert-action:

\[ \delta S_H = \delta S_1 + \delta S_2 + \delta S_3 \quad \leftrightarrow \]

\[
\delta S_H = \int d^n x \sqrt{-g} R_{\mu \nu} \delta g^{\mu \nu} + \int d^n x R(-\frac{1}{2} \sqrt{-g} g_{\alpha \beta} \delta g^{\alpha \beta}) =
\]

\[= \int d^n x \sqrt{-g} (R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu}) \delta g^{\mu \nu} \equiv 0 \]
This is satisfied only if the coefficient of the variation vanishes:

$$\sqrt{-g}(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = 0$$  \hspace{1cm} (A.13)

This is Einstein’s field equations in vacuum.

For the non-vacuum solution, we look at the entire action $S$, including the gravitational action and the matter action $S_M$, given by a matter source in the spacetime. We normalize the sum of the Einstein-Hilbert action and the matter action in a way for the final result to have the desired form,

$$S = \frac{1}{16\pi G} S_H + S_M$$  \hspace{1cm} (A.14)

Under a variation of the gravitational field, the variational coefficients are given by

$$\frac{\delta S}{\delta g^{\mu\nu}}$$  \hspace{1cm} (A.15)

which should vanish for the principle of least action, demanding $\delta S = 0$. For the total action we thus get

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G} \frac{\delta S_H}{\delta g^{\mu\nu}} + \frac{\delta S_M}{\delta g^{\mu\nu}} =$$

$$= \frac{1}{16\pi G} \sqrt{-g} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) + \frac{\delta S_M}{\delta g^{\mu\nu}} = 0$$

$$\Leftrightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -16\pi G \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$$  \hspace{1cm} (A.16)

Now we define an stress-energy tensor $T_{\mu\nu}$ to be

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$$  \hspace{1cm} (A.17)

So that we can rewrite the condition for the principle of least action (A.16) and so receive Einstein’s field equations with zero cosmological constant $\Lambda$,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$  \hspace{1cm} (A.18)

Or equivalently, by using the Einstein tensor:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$  \hspace{1cm} (A.19)

**B  Interior metric of static cosmic string**

For a static, axisymmetric (cylindrically symmetric) metric with Lorentzian signature we can assume the form of the line element to be, given in [12]:

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + e^{2\Psi(r)} d\theta^2 + e^{2\lambda(r)} dz^2$$  \hspace{1cm} (B.1)
with the metric on a matrix form (suppressing the arguments of the functions):

\[
g_{\mu\nu} = \begin{bmatrix}
-\epsilon^{2\nu} & 0 & 0 & 0 \\
0 & \epsilon^{2\lambda} & 0 & 0 \\
0 & 0 & \epsilon^{2\Psi} & 0 \\
0 & 0 & 0 & \epsilon^{2\lambda}
\end{bmatrix}
\]

with inverse metric:

\[
g_{\mu\nu} = \begin{bmatrix}
-\epsilon^{-2\nu} & 0 & 0 & 0 \\
0 & \epsilon^{-2\lambda} & 0 & 0 \\
0 & 0 & \epsilon^{-2\Psi} & 0 \\
0 & 0 & 0 & \epsilon^{-2\lambda}
\end{bmatrix}
\]

Since we have a diagonal metric, all Christoffel symbols with all indices different will vanish, as all terms being partially derivated will have off-diagonal components (which are all zero). Further we note that all non-zero components of the metric are only dependent on the \(r\) coordinate, and thus we must have a partial derivative in order for the symbol not to vanish. Due to the diagonality of the metric, the dummy variable in the expression for the Christoffel symbols always takes on the same index as the symbols contravariant index. Thus, we must at least have one \(r\) index in the non-zero symbols. We are left with all symbols which either have two \(r\) indices and one other, or one \(r\) index and two other but mutually same indices. These non-vanishing Christoffel symbol will be:

\[
\Gamma_{t r} = \frac{1}{2} g^{tt}(\partial_{t} g_{tt}) = \frac{1}{2} (-\epsilon^{-2\nu})(-2\nu')\epsilon^{2\nu} = \nu'
\]

\[
\Gamma_{t t} = \frac{1}{2} g^{rr}(\partial_{t} g_{tt}) = \frac{1}{2} \epsilon^{2\lambda}(2\nu')\epsilon^{2\nu} = \nu'e^{2(\nu'-\lambda)}
\]

\[
\Gamma_{r r} = \frac{1}{2} g^{rr}(\partial_{r} g_{rr}) = \frac{1}{2} \epsilon^{-2\lambda}(2\lambda')\epsilon^{2\lambda} = \lambda'
\]

\[
\Gamma_{\theta \theta} = \frac{1}{2} g^{rr}(\partial_{r} g_{\theta\theta}) = \frac{1}{2} \epsilon^{-2\lambda}(2\Psi'\epsilon^{2\theta}) = -\Psi'e^{2(\Psi-\lambda)}
\]

\[
\Gamma_{\rho \rho} = \frac{1}{2} g^{zz}(\partial_{r} g_{\rho\rho}) = \frac{1}{2} \epsilon^{-2\lambda}(2\lambda'\epsilon^{2\lambda}) = -\lambda'
\]

\[
\Gamma_{r \theta} = \frac{1}{2} g^{\theta\theta}(\partial_{r} g_{\theta\theta}) = \frac{1}{2} \epsilon^{2\Psi}(2\Psi'\epsilon^{2\Phi}) = \Psi'
\]

\[
\Gamma_{r z} = \frac{1}{2} g^{zz}(\partial_{r} g_{zz}) = \frac{1}{2} \epsilon^{2\lambda}(2\lambda'\epsilon^{2\lambda}) = \lambda'
\]

We wish to calculate all the diagonal Ricci tensor components and the Ricci scalar, which we get by:

\[
R = R^t_t + R^r_r + R^\theta_\theta + R^z_z
\]

We calculate each of the diagonal Ricci tensor components in their \((0, 2)\)-type tensor forms in steps. We note firstly that all Riemann tensor components with all indices being the same will all vanish, from the form of the Riemann tensor:
\[ R^r_{ttt} = \partial_r \Gamma^r_{tt} - \partial_t \Gamma^r_{rt} + \Gamma^r_{\alpha \beta} \Gamma^\alpha_\gamma \Gamma^\beta_\lambda - \Gamma^r_{t \alpha} \Gamma^\alpha_\lambda \Gamma^\beta_\gamma \]
\[ = \partial_r \Gamma^r_{tt} + \Gamma^r_{rr} \Gamma^r_{tt} - \Gamma^r_{tt} \Gamma^r_{rt} = \]
\[ = \partial_r (\nu' e^{2(\nu - \lambda)}) + \lambda' \nu' e^{2(\nu - \lambda)} - \nu'^2 e^{2(\nu - \lambda)} = \]
\[ = \nu'' e^{2(\nu - \lambda)} + \nu'^2 (\nu' - \lambda') e^{2(\nu - \lambda)} + \lambda' \nu' e^{2(\nu - \lambda)} - \nu'^2 e^{2(\nu - \lambda)} = \]
\[ = e^{2(\nu - \lambda)} (\nu'' - \nu'^2 - \lambda' \nu') \]
\[ R^\theta_{t \theta t} = \partial_\theta \Gamma^\theta_{t \theta} - \partial_t \Gamma^\theta_{t \theta} + \Gamma^\theta_{\alpha \beta} \Gamma^\alpha_\gamma \Gamma^\beta_\lambda - \Gamma^\theta_{t \alpha} \Gamma^\alpha_\lambda \Gamma^\beta_\gamma \]
\[ = \Gamma^\theta_{\theta \theta} \Gamma^\theta_{t \theta} = \]
\[ = \Psi' \nu' e^{2(\nu - \lambda)} \]
\[ R^z_{z z t} = \partial_z \Gamma^z_{z t} - \partial_t \Gamma^z_{z t} + \Gamma^z_{\alpha \beta} \Gamma^\alpha_\gamma \Gamma^\beta_\lambda - \Gamma^z_{z \alpha} \Gamma^\alpha_\lambda \Gamma^\beta_\gamma \]
\[ = \Gamma^z_{z z} \Gamma^z_{t t} = \]
\[ = \lambda' \nu' e^{2(\nu - \lambda)} \]
\[ \Rightarrow R_{tt} = R^t_{t t t} + R^r_{t r t} + R^\theta_{t \theta t} + R^z_{z z t} = \]
\[ = e^{2(\nu - \lambda)} (\nu'' + \nu'^2 - \lambda' \nu' + \Psi' \nu' + \lambda' \nu') = \]
\[ = e^{2(\nu - \lambda)} (\nu'' + \nu'^2 + \Psi' \nu') \]

2) \[ R^t_{r r r} = \partial_r \Gamma^t_{r r} - \partial_r \Gamma^t_{r t} + \Gamma^t_{\alpha \beta} \Gamma^\alpha_\gamma \Gamma^\beta_\lambda - \Gamma^t_{r \alpha} \Gamma^\alpha_\lambda \Gamma^\beta_\gamma \]
\[ = \partial_t \Gamma^r_{tt} + \Gamma^t_{tr} \Gamma^r_{tt} - \Gamma^r_{tt} \Gamma^t_{rt} = \]
\[ = -\partial_r (\nu' \nu' - \nu'^2) = -\nu'' + \lambda' \nu' - \nu'^2 \]
\[ R^\theta_{r \theta r} = \partial_\theta \Gamma^\theta_{r r} - \partial_r \Gamma^\theta_{r \theta} + \Gamma^\theta_{\alpha \beta} \Gamma^\alpha_\gamma \Gamma^\beta_\lambda - \Gamma^\theta_{r \alpha} \Gamma^\alpha_\lambda \Gamma^\beta_\gamma \]
\[ = -\partial_\theta (\Psi' \nu' - \Psi'^2) = -\Psi'' + \Psi' \nu' - \Psi'^2 \]
\[ R^z_{z r z} = \partial_z \Gamma^z_{r z} - \partial_r \Gamma^z_{r z} + \Gamma^z_{\alpha \beta} \Gamma^\alpha_\gamma \Gamma^\beta_\lambda - \Gamma^z_{z \alpha} \Gamma^\alpha_\lambda \Gamma^\beta_\gamma \]
\[ = -\partial_r (\lambda' \nu' - \lambda'^2) = -\lambda'' \]
\[ \Rightarrow R_{r r} = R^t_{t r t} + R^r_{r r r} + R^\theta_{r \theta r} + R^z_{z r z} = \]
\[ = -\nu'' + \lambda' \nu' - \nu'^2 - \Psi'' + \Psi' \nu' - \Psi'^2 - \lambda' \]
3) 

\[ R^t_{\theta t \theta} = \partial_r \Gamma^r_{\theta 0} - \partial_\theta \Gamma^r_{\theta 0} + \Gamma^t_{\lambda \lambda} \Gamma^\lambda_{\theta 0} - \Gamma^t_{\theta \lambda} \Gamma^\lambda_{\theta 0} = \]
\[ = \Gamma^t_{\theta r} \Gamma^r_{\theta 0} = \]
\[ = -\nu' \Psi e^{2(\Psi - \lambda)} \]

\[ R^r_{r \theta \theta} = \partial_\theta \Gamma^r_{\theta 0} - \partial_\theta \Gamma^r_{\theta 0} + \Gamma^r_{\lambda \lambda} \Gamma^\lambda_{\theta 0} - \Gamma^r_{\theta \lambda} \Gamma^\lambda_{\theta 0} = \]
\[ = \partial_\theta (\psi e^{2(\Psi - \lambda)}) + \lambda' (-\psi e^{2(\Psi - \lambda)}) + \psi'^2 e^{2(\Psi - \lambda)} = \]
\[ = e^{2(\Psi - \lambda)} (\psi'' - \psi' \psi'') + \psi'^2 = \]
\[ = e^{2(\Psi - \lambda)} (\psi'' - \psi' \psi'^2 + \psi' \lambda') \]

\[ R^z_{\theta z \theta} = \partial_\theta \Gamma^z_{\theta 0} - \partial_\theta \Gamma^z_{\theta 0} + \Gamma^z_{\lambda \lambda} \Gamma^\lambda_{\theta 0} - \Gamma^z_{\theta \lambda} \Gamma^\lambda_{\theta 0} = \]
\[ = \Gamma^z_{\theta r} \Gamma^r_{\theta 0} = \]
\[ = \lambda' (-\psi e^{2(\Psi - \lambda)}) \]

\[ \Rightarrow R_{00} = R^t_{\theta t \theta} + R^r_{r \theta \theta} + R^0_{\theta 0 \theta} + R^z_{\theta z \theta} = \]
\[ = e^{2(\Psi - \lambda)} (\psi'' - \psi' \psi'^2 + \psi' \lambda') \]

4) 

\[ R^t_{ztz} = \partial_r \Gamma^r_{zt} - \partial_\lambda \Gamma^\lambda_{zt} + \Gamma^t_{\lambda \lambda} \Gamma^\lambda_{zt} - \Gamma^t_{\lambda \lambda} \Gamma^\lambda_{zt} = \]
\[ = \Gamma^t_{zt} \Gamma^r_{zt} = \]
\[ = -\nu' \lambda' \]

\[ R^r_{rzz} = \partial_r \Gamma^r_{zz} - \partial_\lambda \Gamma^\lambda_{zz} + \Gamma^r_{\lambda \lambda} \Gamma^\lambda_{zz} - \Gamma^r_{\lambda \lambda} \Gamma^\lambda_{zz} = \]
\[ = \partial_\lambda \Gamma^\lambda_{zz} + \Gamma^r_{\lambda \lambda} \Gamma^\lambda_{zz} - \Gamma^r_{\lambda \lambda} \Gamma^\lambda_{zz} = \]
\[ = \partial_r (-\lambda') + \lambda' (-\lambda') - (-\lambda') \lambda' = \]
\[ = -\lambda'' \]

\[ R^z_{\theta zz} = \partial_\theta \Gamma^z_{\theta z} - \partial_\theta \Gamma^z_{\theta z} + \Gamma^z_{\lambda \lambda} \Gamma^\lambda_{\theta z} - \Gamma^z_{\theta \lambda} \Gamma^\lambda_{\theta z} = \]
\[ = \Gamma^z_{\theta r} \Gamma^r_{\theta z} = \]
\[ = -\lambda' \Psi' \]

\[ \Rightarrow R_{zz} = R^t_{ztz} + R^r_{rzz} + R^\theta_{\theta zz} + R^z_{zzz} = \]
\[ = -\nu' \lambda' - \lambda'' - \lambda' \Psi' \]

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And finally this allows us to calculate the Ricci scalar:

\[
R = R^t_t + R^r_r + R^\theta_\theta + R^z_z = \\
= g^{tt} R_{tt} + g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} + g^{zz} R_{zz} = \\
= g^{tt} R_{tt} + g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} + g^{zz} R_{zz} = \\
= (-e^{-2\nu})e^{2(\nu-\lambda)}[\nu'' + \nu'^2 + \Psi'\nu''] + e^{-2\nu}[\nu'' + \nu'\lambda' - \nu'^2 + \Psi'\lambda' - \Psi'^2 - \lambda''] + \\
+ e^{-2\Psi} e^{2(\Psi-\lambda)}[\nu'\Psi' - \Psi'' - \Psi'^2] + e^{-2\lambda}[\nu'\Psi' - \lambda' + \lambda'\Psi'] = \\
= e^{-2\lambda}[\nu'' + \nu'^2 + \Psi'\nu' + \Psi' + \nu'\lambda' - \Psi'^2 + \Psi'' + \lambda''] = -2e^{-2\lambda}[\nu'' + \nu'^2 + \nu'\Psi' + \Psi'' + \Psi'^2 + \lambda'']
\]

Now we are ready to apply all in Einstein’s field equations. As we only have diagonal entries in the stress-energy tensor, we only consider the diagonal equations.

1) \(tt\)-component:

\[
R_{tt} - \frac{1}{2} R g_{tt} = 8\pi G T_{tt} \\
\Rightarrow g^{tt} R_{tt} - \frac{1}{2} R = 8\pi G T_{tt} \\
\Leftrightarrow \\
- e^{-2\lambda}[\nu'' + \nu'^2 + \Psi'\nu'] - \frac{1}{2}(e^{-2\lambda})[\nu'' + \nu'^2 + \nu'\Psi' + \Psi'' + \Psi'^2 + \lambda''] = -8\pi G\kappa \\
\Leftrightarrow \\
e^{-2\lambda}[\nu'' + \nu'^2 + \Psi'\nu' - \nu'\lambda' + \lambda'\Psi' - \Psi'' - \Psi'^2 - \lambda''] = -8\pi G\kappa \\
\Leftrightarrow \\
e^{-\lambda}[\Psi'' + \Psi'^2 + \lambda''] = -8\pi G\kappa \tag{B.3}
\]

2) \(rr\)-component:

\[
R_{rr} - \frac{1}{2} R g_{rr} = 0 \\
\Rightarrow g^{rr} R_{rr} - \frac{1}{2} R = 0 \\
\Leftrightarrow \\
e^{-2\lambda}[\nu'' + \nu'\lambda' - \nu'^2 - \Psi'' + \Psi'\lambda' - \Psi'^2 - \lambda''] - \\
- \frac{1}{2}(e^{-2\lambda})[\nu'' + \nu'^2 + \nu'\Psi' + \Psi'' + \Psi'^2 + \lambda''] = 0 \\
\Leftrightarrow \\
e^{-2\lambda}[\nu'' + \nu'\lambda' - \nu'^2 - \Psi'' + \Psi'\lambda' - \Psi'^2 - \lambda''] + \\
+ \nu'\lambda' + \lambda'\Psi' + \Psi'\lambda' + \Psi'^2 + \lambda''] = 0 \\
\Leftrightarrow \\
e^{-2\lambda}[\nu'\Psi' + \nu'\lambda' + \Psi'\lambda'] = 0 \tag{B.4}
\]
3) $\theta\theta$-component:

$$R_{\theta\theta} - \frac{1}{2} R g_{\theta\theta} = 0$$

$$\Rightarrow g^{\theta\theta} R_{\theta\theta} - \frac{1}{2} R = 0$$

$$\Leftrightarrow e^{-2\Psi} e^{2(\Psi - \lambda)} [-\nu'\Psi' - \Psi'' - \Psi] - \frac{1}{2} (-2e^{-2\lambda})[\nu'' + \nu'^2 + \nu'\Psi' + \Psi'' + \Psi^2 + \lambda'']$$

$$\Leftrightarrow e^{-2\lambda} [-\nu'\Psi' - \Psi'' + \nu'' + \nu'^2 + \mu, \Psi' + \Psi'' + \Psi^2 + \lambda''] = 0$$

$$\Leftrightarrow e^{-2\lambda} [\nu'' + \nu'^2 + \lambda''] = 0 \quad \text{(B.5)}$$

4) $zz$-component:

$$R_{zz} - \frac{1}{2} R g_{zz} = 8\pi G T_{zz}$$

$$\Rightarrow g^{zz} R_{zz} - \frac{1}{2} R = 8\pi G T_{zz}$$

$$\Leftrightarrow e^{-2\lambda} [-\nu'\lambda' - \lambda'\Psi' + \Psi'' + \nu'^2 + \nu'\Psi' + \Psi'' + \Psi^2 + \lambda''] = -8\pi G \kappa$$

$$\Leftrightarrow e^{-2\lambda} [-\nu'\lambda' - \lambda'\Psi' + \Psi'' + \nu'^2 + \nu'\Psi' + \Psi'' + \Psi^2 + \lambda''] = -8\pi G \kappa$$

$$\Leftrightarrow e^{-2\lambda} [\nu'' + \nu'^2 - \nu'\lambda' + \nu'\Psi' - \lambda'\Psi' + \Psi'' + \Psi^2] = -8\pi G \kappa \quad \text{(B.6)}$$

Equations (B.3), (B.4), (B.5), (B.6) are the four independent Einstein’s field equations for a static cosmic string.

### C Verification of the density operator (4.13)

We wish to prove that (4.13) solves (4.12) with the specific interaction (4.8). Firstly, we expand the unitary operator acting on the density operator of the entire density operator,

$$U_1(\rho_s \otimes \rho_q) = \sum_{\alpha'} \sum_{\beta'} |\alpha' + \beta'\rangle \langle \beta'| (\langle s | \langle s | q \rangle \langle q |) =$$

$$= \sum_{\alpha'} \sum_{\beta'} \langle s | \langle s | q | \alpha' + \beta' \rangle \langle \beta'| \rangle$$
and acting with the Hermitian conjugate $U_1^\dagger$:

$$U_1 (\rho_s \otimes \rho_q) U_1^\dagger = \sum_{\alpha'} \sum_{\beta'} \sum_{\alpha} \sum_{\beta} (\alpha' \mid s \rangle \langle \beta' \mid q \rangle \langle \alpha' + \beta' \mid \alpha' \rangle \langle \alpha + \beta \mid \beta \rangle =$$

$$= \sum_{\alpha'} \sum_{\beta'} \sum_{\alpha} \sum_{\beta} (\alpha' \mid s \rangle \langle \beta' \mid q \rangle \langle \alpha \mid q \rangle \langle \beta \mid \alpha' + \beta \rangle \langle \alpha + \beta \mid \beta \rangle , \quad (C.1)$$

where in the last line we have written the total density operator as the direct product of the two different single-particle states: the first denoting the particle $s$, the other denoting space of particle $q$. Now we wish to take the partial trace of this total state over the $q$-particle states. Let us look at how this is defined.

Having a sum of the form

$$\sum_{i,j} A_{ij} \left| a^{(i)} \right\rangle \left\langle a^{(j)} \right| , \quad (C.2)$$

and the sum is over the set $V$ of basis elements of the vectors $|a^{(i)}\rangle$, then the trace over $V$ is defined to be

$$Tr_V \left( \sum_{i,j} A_{ij} \left| a^{(i)} \right\rangle \left\langle a^{(j)} \right| \right) = \sum_k A_{kk} . \quad (C.3)$$

Now we can take the partial trace over the set of basis of the second particle state $U \supseteq \beta, \beta'$ of (C.1),

$$Tr_U \left( U_1 (\rho_s \otimes \rho_q) U_1^\dagger \right) = \sum_{\alpha'} \sum_{\alpha} \sum_{\beta} \langle \alpha' \mid s \rangle \langle \beta \mid q \rangle \langle s \mid q \rangle \langle \alpha \mid \beta \rangle \langle \alpha + \beta \rangle =$$

$$= (0 \mid s \rangle \langle q \rangle \langle 0 \mid q \rangle \langle 0 \mid q \rangle \langle 0 \mid q \rangle \langle 0 \mid q \rangle \langle 0 \rangle \langle 1 \rangle +$$

$$+ \langle 0 \mid s \rangle \langle q \rangle \langle 0 \rangle \langle 0 \rangle \langle 1 \rangle \langle 1 \rangle +$$

$$+ \langle 0 \mid s \rangle \langle q \rangle \langle 1 \rangle \langle 1 \rangle \langle 0 \rangle +$$

$$+ \langle 1 \mid s \rangle \langle q \rangle \langle 0 \rangle \langle 0 \rangle \langle 0 \rangle \langle 1 \rangle +$$

$$+ \langle 1 \mid s \rangle \langle q \rangle \langle 1 \rangle \langle 1 \rangle \langle 1 \rangle +$$

$$+ \langle 1 \mid s \rangle \langle q \rangle \langle 1 \rangle \langle 1 \rangle \langle 0 \rangle \langle 0 \rangle , \quad (C.4)$$

and now let us for the CTC ingoing state $|q\rangle$ assume the general superposition (normalized) state

$$|q\rangle = d_1 |0\rangle + d_2 |1\rangle , \quad |d_1|^2 + |d_2|^2 = 1 , \quad (C.5)$$

and replacing this in (C.4), we get

$$Tr_U \left( U_1 (\rho_s \otimes \rho_q) U_1^\dagger \right) = |d_1|^2 \langle 0 \mid s \rangle \langle 0 \mid q \rangle \langle 0 \rangle \langle 0 \rangle +$$

$$+ |d_2|^2 \langle 0 \mid s \rangle \langle 0 \rangle \langle 1 \rangle \langle 1 \rangle +$$

$$+ |d_1|^2 \langle 1 \mid s \rangle \langle 0 \rangle \langle 1 \rangle \langle 0 \rangle +$$

$$+ |d_2|^2 \langle 1 \mid s \rangle \langle 1 \rangle \langle 1 \rangle \langle 1 \rangle +$$

$$+ |d_1|^2 \langle 1 \mid q \rangle \langle 0 \rangle \langle 0 \rangle \langle 1 \rangle +$$

$$+ |d_2|^2 \langle 1 \mid q \rangle \langle 1 \rangle \langle 1 \rangle \langle 0 \rangle =$$

$$= \left(|0 \rangle \langle 1 |s|^2 |d_1|^2 \right) \langle 0 \rangle \langle 0 \rangle +$$

$$+ \left(|0 \rangle \langle 1 |s|^2 |d_2|^2 \right) \langle 1 \rangle \langle 1 \rangle +$$

$$+ \left(|1 \rangle \langle s |0|^2 |d_1|^2 \right) \langle 0 \rangle \langle 1 \rangle +$$

$$+ \left(|1 \rangle \langle s |0|^2 |d_2|^2 \right) \langle 1 \rangle \langle 1 \rangle +$$

$$+ \left(|0 \rangle \langle s |1|^2 |d_1|^2 \right) \langle 1 \rangle \langle 0 \rangle +$$

$$+ \left(|0 \rangle \langle s |1|^2 |d_2|^2 \right) \langle 1 \rangle \langle 1 \rangle , \quad (C.6)$$

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and we wish to equate this with

$$\rho_q = |q\rangle \langle q| = |d_1|^2 |0\rangle \langle 0| + d_2^* d_2 |1\rangle \langle 1| + |d_2|^2 |1\rangle \langle 1|,$$

(C.7)

and for this to be equal to (C.6) we need the coefficients of the states to be necessarily the same, since the states are linearly independent. This gives us two independent equations,

$$|\langle 0 | s \rangle|^2 |d_1|^2 + |\langle 1 | s \rangle|^2 |d_2^*|^2 = |d_1|^2,$$

(C.8)

$$|\langle 1 | s \rangle|^2 |d_2|^2 + |\langle 0 | s \rangle|^2 |d_1|^2 |1\rangle \langle 1| = d_1 d_2^*,$$

(C.9)

We have the normalization condition for $|s\rangle$

$$|\langle 0 | s \rangle|^2 + |\langle 1 | s \rangle|^2 = 1,$$

(C.10)

which can be replaced in (C.8) to get

$$|\langle 1 | s \rangle|^2 |d_2|^2 = |\langle 1 | s \rangle|^2 |d_1|^2,$$

$$\Rightarrow |d_1|^2 = |d_2|^2 = \frac{1}{2},$$

(C.11)

where in the last step we have used the normalization condition (C.5).

Looking at equation (C.9), with replacing the result (C.11),

$$\langle 1 | s \rangle \langle s | 0 \rangle + (\langle 1 | s \rangle \langle s | 0 \rangle)^* = 2d_1 d_2^*,$$

$$\Leftrightarrow \text{Re} [\langle 1 | s \rangle \langle s | 0 \rangle] = d_1 d_2^*,$$

(C.12)

which also gives

$$d_1^* d_2 = (d_1 d_2^*)^* = \text{Re} [\langle 1 | s \rangle \langle s | 0 \rangle].$$

(C.13)

Combining these solutions, the density operator $\rho_q$ from (C.7) which satisfies (4.12) is

$$\rho_q = \frac{1}{2} (|0\rangle \langle 0| + |1\rangle \langle 1|) + \text{Re} [\langle 1 | s \rangle \langle s | 0 \rangle] (|0\rangle \langle 0| + |1\rangle \langle 1|) + |d_2|^2 |1\rangle \langle 1|,$$

(C.14)

which is exactly (4.13), using the completeness relation $\mathbb{I} = |0\rangle \langle 0| + |1\rangle \langle 1|$ for the single-particle state.
References


