Coefficients and zeros of mixed characteristic polynomials

SAMUEL ZACKRISSON
Coefficients and zeros of mixed characteristic polynomials

SAMUEL ZACKRISSON
Acknowledgements

Thanks, mum. This is for you.

Big thanks to Petter, the all-star of thesis advising, for many thought-provoking discussions.

Shoutout to Hanna for being a get-it-done role model.
Abstract

The mixed characteristic polynomial (MCP) was introduced in the papers of Marcus, Spielman and Srivastava from 2013 on Ramanujan graphs and the Kadison-Singer conjecture. Several known results and open problems can be formulated in terms of MCPs. The proofs of Marcus, Spielman and Srivastava involve bounding the roots of certain MCPs. Gurvits’ generalization of van der Waerden’s permanent conjecture bounds the constant term of MCPs using the capacity of an underlying polynomial.

This thesis surveys selected results for MCPs. A counterexample to the Holens-Doković conjecture, due to Wanless, is discussed in the context of MCPs. It is used to show how a sequence of MCP coefficients is not monotone and how the roots of associated Laguerre polynomials do not always majorize those of other MCPs. Finally, we prove an analogue of the root bound in the proof of the Kadison-Singer conjecture. It applies to product polynomials of doubly stochastic matrices through classical results in graph theory due to Godsil, Mohar, Heilmann and Lieb.
Sammanfattning

Koefficienter och nollställen hos blandade karaktäristiska polynom


# Contents

1 Introduction 3

2 Setting the scene: Stable polynomials 4
   2.1 Definitions and examples 4
   2.2 Preserving stability 5
   2.3 Elementary symmetric polynomials, operators and an inequality 8
   2.4 Homogeneity and double stochasticity 10
   2.5 A probabilistic perspective 11

3 Gurvits’ theorem 14
   3.1 Statement 14
   3.2 Proof of Gurvits’ Theorem 15
   3.3 Tightness 16
   3.4 An extension for doubly stochastic polynomials 18

4 Mixed characteristic polynomials 20
   4.1 Definition and properties 20
   4.2 A combinatorial MCP example 21
   4.3 A resampling operator 22
   4.4 Monotonicity of coefficients of certain MCPs 25
   4.5 An induced map of MCPs 29

5 Graphs, matrices and stable polynomials 31
   5.1 Matrices and product polynomials 31
      5.1.1 Permanents 31
      5.1.2 Van der Waerden’s permanent conjecture 32
      5.1.3 Wanless’ counterexample to the Holens-Doković conjecture 33
   5.2 Stable polynomials from graphs 34
      5.2.1 Definitions 34
      5.2.2 Matching polynomials of bipartite graphs are basically MCPs 35
   5.3 Root bounds on graph MCPs 36
      5.3.1 Some properties of matching polynomials 36
      5.3.2 Real-rootedness and divisibility of matching polynomials 38
      5.3.3 From divisibility to root bounds 42

6 Future work 45

7 Index of notation 46

A Appendix: Wanless’ counterexample MCP 49
1 Introduction

Stable polynomials have a long history in mathematics, dating back to attempts on the Riemann hypothesis. Interest in them, in the subject of combinatorics, has increased considerably in the last decades. A prominent article on the subject was published in 2002 by Choe, Oxley, Sokal and Wagner [1], relating stable polynomials to matroid theory.

Since then three results hint at the success of the recent research effort into stable polynomials:

In 2006, Leonid Gurvits gave a vast generalization of van der Waerden’s permanent conjecture [2]. The proof is surprisingly simple, especially compared to the original fairly technical proofs. Gurvits’ theorem has applications in matrix theory and theoretical computer science.

In a sequence of articles in 2008 and 2009, Julius Borcea and Petter Brändén completely characterized the linear operators preserving stability [3] [4]. Chaining stability-preservers is a common technique of proving stability and their work lays the entire toolbox of stability preservers bare for mathematicians to use.

Finally, as recently as 2013, Adam Marcus, Daniel Spielman and Nikhil Srivastava used the theory of stable polynomials to prove the long-standing Kadison-Singer conjecture [5] and show that there exist infinite families of Ramanujan graphs of any given degree [6]. In 2015 they extended this to Ramanujan graphs of any given size [7].

Marcus, Spielman and Srivastava introduce the mixed characteristic polynomial [5]. Their key result lies in proving bounds on its roots, a difficult exercise. Gurvits’ theorem can also be seen as an inequality on the coefficients of the mixed characteristic polynomials. However, there are still many open questions about these polynomials.

Doubly stochastic product polynomials have mixed characteristic polynomials that coincide with matching polynomials of certain weighted bipartite graphs. This allows one to use classical results on matching polynomials, in particular related to an article from 1971 by Ole Heilman and Elliott Lieb [8] and the works of Godsil and Mohar [9] [10], to infer properties of the mixed characteristic polynomials of doubly stochastic product polynomials.
2 Setting the scene: Stable polynomials

2.1 Definitions and examples

Consider the spaces $\mathbb{C}[z_1, \ldots, z_m]$ and $\mathbb{R}[x_1, \ldots, x_m]$ of multivariate polynomials with coefficients from $\mathbb{C}$ or $\mathbb{R}$. Here’s one:

$$R(x_1, x_2, x_3, x_4) = 1 + 2x_1x_2 + x_1^3 x_4.$$ 

We place restrictions on the zeros of our polynomials by defining stability, a notion borrowed from control theory.

**Definition 2.1.** For a subset $\Omega \subseteq \mathbb{C}^m$, a polynomial $P \in \mathbb{C}[z_1, \ldots, z_m]$ is $\Omega$-stable if $P(z) \neq 0$ for all $z \in \Omega$. Let $H_+ = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$, the upper half-plane of $\mathbb{C}$. A $H_+^m$-stable polynomial $P$ is called stable and if $P \in \mathbb{R}[x_1, \ldots, x_m]$, we call $P$ real stable.

The real stable polynomials are the main objects of study in this thesis, although we will often bring out $\mathbb{C}[z_1, \ldots, z_m]$ in this particular chapter.

Reconsidering the polynomial $R$, it’s an unfortunate example - it has a zero $(i, i, i, i) \in H_+^4$ and is therefore not stable.

The simplest real stable polynomials are found among the univariate polynomials. The stable polynomials of $\mathbb{R}[x]$ are precisely the real-rooted ones. A polynomial $P \in \mathbb{R}[x]$ is stable if no root has positive imaginary part. $P$ has real coefficients, so all complex roots come in conjugate pairs and hence no root has negative imaginary part either.

For a matrix $A \in \mathbb{R}^{d \times m}$ with non-negative entries, we define its product polynomial

$$P_A = \prod_{i \in [d]} (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{im} x_m), \quad (1)$$

where $[d] = \{1, 2, \ldots, d\}$. The product polynomials are stable. Indeed, $P_A$ has a zero at $x$ if and only if $\sum_{j \in [m]} a_{ij} x_j = 0$ for some $i$. If $x \in H_+^m$, then $\text{Im} \left( \sum_{j \in [m]} a_{ij} x_j \right) > 0$ for all $i$ and hence $P_A(x) \neq 0$.

As a final example let $P = \det \left( A_0 + \sum_{j \in [m]} x_j A_j \right)$ for hermitian matrices $A_j \in \mathbb{R}^{m \times m}$ where $A_j$ is positive semidefinite for $j \in [m]$. This is a determinantal polynomial. It is either stable or identically zero as we will see in the next section.
2.2 Preserving stability

The set of stable polynomials is surprisingly robust. There are many linear transformations that preserve stability. These transformations were completely classified as part of the Polyá-Schur program, culminating in two seminal papers by Borcea and Brändén \[3\] \[4\].

They have several important closure properties. For example, scaling or multiplying stable polynomials gives new stable polynomials. This will be used throughout the thesis without reference. Another powerful property is captured by Hurwitz’ theorem.

**Theorem 2.2** (Hurwitz’ theorem). Let \( \Omega \subseteq \mathbb{C}^m \) be a connected, open set, and let \( \{f_k\} \) be a sequence of \( \Omega \)-stable analytic functions \( f_k : \mathbb{C}^m \to \mathbb{C} \) that converges to \( f \) uniformly on compact subsets of \( \mathbb{C}^m \). Then \( f \) is either \( \Omega \)-stable or identically zero on \( \Omega \).

A similar formulation and a proof can be found as Corollary 2.6 in Chapter 7 of Conway’s classic textbook \[11\].

Hurwitz’ theorem allows us to prove stability of a set of polynomials by proving it for a dense subset and then taking appropriate limits. To illustrate this we prove stability of the determinantal polynomials.

**Proposition 2.3.** Determinantal polynomials \( P = \det \left( A_0 + \sum_{j \in [m]} x_j A_j \right) \) are stable or identically zero.

**Proof.** \[12\]. Assume that the \( A_j, j \in [m] \), are positive definite. Let \( x \in H^m_+ \), so that \( x_j = a_j + ib_j \) for reals \( a_j, b_j \) where \( y_j > 0 \). Then

\[
P(z) = \det \left( A_0 + \sum_{j \in [m]} a_j A_j + i \cdot \sum_{j \in [m]} b_j A_j \right) = \det \left( Q + iR \right), \tag{2}
\]

for hermitian \( Q, R \) where \( R = \sum_{j \in [m]} b_j A_j \) is a conical combination of positive definite matrices and is hence positive definite with an invertible square root \( R^{\frac{1}{2}} \).

\[
P(z) = \det \left( Q + iR \right) = \det \left( iI + R^{-\frac{1}{2}} \right) \cdot \det \left( R^{-\frac{1}{2}} \right). \tag{3}
\]

Since the hermitian matrix \( R^{-\frac{1}{2}} \) only has real eigenvalues, \( P(x) \neq 0 \) for \( x \in H^m_+ \). Hence \( P \) is stable.

If some \( A_j, j \in [m] \), is not positive definite then we can take sequences of invertible matrices \( \{A^k_j\}_k \) for which \( \lim_{k \to \infty} A^k_j \to A_j \). The polynomial \( P_k = \det \left( A_0 + \sum_{j \in [m]} z_j A^k_j \right) \) is stable and as \( k \to \infty, P_k \to P \) uniformly on compact subsets of \( \Omega \). By Theorem 2.2, \( P \) is either stable or identically zero. \( \square \)
We will be using several results on stability-preserving operators, the most general of which is found as Theorem 1.2 in [3]. Take $\kappa \in \mathbb{N}^m$ and let $\mathbb{R}_\kappa[x_1, \ldots, x_m]$ be the polynomials $P$ of $\mathbb{R}[x_1, \ldots, x_m]$ for which the degree of $x_j$ is bounded by $\kappa_j$ for $j \in [m]$: $\deg_{x_j}(P) \leq \kappa_j$. Given an operator $T : \mathbb{R}_\kappa[x_1, \ldots, x_m] \rightarrow \mathbb{R}[x_1, \ldots, x_m]$, define its symbol $G_T$:

$$G_T(x, w) = T \left( \prod_{j \in [m]} (x_j + w_j)^{\kappa_j} \right) \in \mathbb{R}[x_1, \ldots, x_m, w_1, \ldots, w_m].$$ (4)

Note that $\prod_{j \in [m]} (x_j + w_j)^{\kappa_j}$ is $H_{2m}^2$-stable.

The stability-preserving properties of $T$ are essentially characterized by its symbol:

**Theorem 2.4.** A linear operator $T : \mathbb{R}_\kappa[x_1, \ldots, x_m] \rightarrow \mathbb{R}[x_1, \ldots, x_m]$ preserves stability if and only if one of the following holds true.

(i) The image of $T$ is at most two dimensional and of the form

$$T(P) = \alpha(P)Q + \beta(P)R$$ (5)

for some linear functionals $\alpha, \beta : \mathbb{R}_\kappa[x_1, \ldots, x_m] \rightarrow \mathbb{R}$ and real stable $Q, R$ such that $R + iQ$ is stable.

(ii) $G_T(x, w)$ is real stable.

(iii) $G_T(x, -w)$ is real stable.

From this we can establish a series of stability-preserving operations, the most important (to us) listed below.

**Lemma 2.5.** Let $P \in \mathbb{R}[x_1, \ldots, x_m]$ be stable of degree $d_j$ in the variable $x_j$. For any $j \in [m]$,

(i) $P(x_1, \ldots, x_{j-1}, \zeta, x_{j+1}, \ldots, x_m)$ is stable or identically zero for all $\zeta \in \mathbb{C}$ with \(\text{Im}(\zeta) \geq 0\);

(ii) $P(x_1, \ldots, x_{j-1}, \lambda x_j, x_{j+1}, \ldots, x_m)$ is stable for all $\lambda > 0$;

(iii) $x_j^{d_j} P(x_1, \ldots, x_{j-1}, -x_j^{-1}, x_{j+1}, \ldots, x_m)$ is stable;

(iv) $P(x_1, \ldots, x_{j-1}, x_k, x_{j+1}, \ldots, x_m)$ is stable;

(v) $\frac{\partial P}{\partial x_j}$ is stable if $d_j \neq 0$ and identically zero if $d_j = 0$;

(vi) $\left( \alpha + \beta t - \frac{\partial}{\partial x_j} \right) P \in \mathbb{R}[x_1, \ldots, x_m, t]$ is stable for $\alpha, \beta \geq 0$.

All of these immediately follow from Theorem 2.4 but it is illustrative to use a variety of techniques.
Proof. It suffices to consider $j = 1$.

(i): Let $Q_\zeta(x_2, \ldots, x_m) = P(\zeta, x_2, \ldots, x_m) \in \mathbb{C}[x_2, \ldots, x_m]$. Let $w \in H_+^{m-1}$ and $\text{Im}(\zeta) > 0$. Then $Q_\zeta(w) = P(\zeta, w) \neq 0$ as $(\zeta, w) \in H_+^m$ and $P$ is stable, and hence $Q_\zeta$ is stable. If $\text{Im}(\zeta) = 0$, we can again employ Hurwitz’ therem. \{Q_{\zeta+i/k}\}_k$ is a sequence of polynomials converging uniformly to $Q_\zeta$ on compact subsets of $H_+^m$, and by Theorem 2.2 $Q_\zeta$ is stable or identically zero.

(ii): $(\lambda x_1, x_2, \ldots, x_m) \in H_+^m$ if and only if $(x_1, \ldots, x_m) \in H_+^m$, so stability of $P$ is equivalent to stability of $P(\lambda x_1, x_2, \ldots, x_m)$.

(iii): We apply Theorem 2.4. The symbol of the transformation $T$ mapping $P \mapsto x_1^{d_1} P(-x_1^{-1}, x_2, \ldots, x_m)$ is

$$G_T(x, w) = (x_1 w_1 - 1)^{d_1} \prod_{j=2}^m (x_j + w_j)^{d_j},$$

which is zero whenever some factor is zero. If $(x, w) \in H_+^{2m}$ then $\text{Im}(x_j + w_j) > 0$ and hence the latter factors are nonvanishing on $H_+^{2m}$. The factor $(x_1 w_1 - 1)$ is zero only if

$$0 = \arg 1 = \arg(1 - w_1) = \arg x_1 + \arg w_1 \mod 2\pi.$$  

When $(x, w) \in H_+^{2m}$, $0 < \arg x_1, w_1 < \pi$ and their argument sum is never zero, hence the symbol is real stable and so is $TP$.

(iv): $P$ is nonvanishing on $H_+^m$, and hence in particular on $H' = \{w \in H_+^m: w_1 = w_k\}$. It follows that $0 \notin P(H')$, which is precisely the image of $H_+^{m-1}$ of the polynomial in $(iv)$.

(v): If $d_1 = 0$ then $\frac{\partial P}{\partial x_1}$ is identically zero. If $d_1 > 0$ then $P$ can be written as $P(x) = \sum_{k=0}^{d_1} x_1^k q_k(x_2, \ldots, x_m)$ for polynomials $q_k \in \mathbb{R}[x_2, \ldots, x_m]$, where $q_{d_1}$ is not identically zero. The sequence of polynomials $P_n(x_1, \ldots, x_m) = n^{-d_1} P(nx_1, x_2, \ldots, x_m)$ converge to the polynomial $x_1^{d_1} q_{d_1}(x_2, \ldots, x_m)$ uniformly on compact subsets of $\mathbb{C}^m$. Theorem 2.2 applies and since $x_1^{d_1} q_{d_1}$ is not identically zero then it must be stable, hence $q_{d_1}$ is stable.

Let $Q_w(x_1) = P(x_1, w_1, \ldots, w_{m-1})$ for $w \in H_+^{m-1}$. Since $d_1 > 0$ and $q_{d_1}(w) \neq 0$, $Q_w$ is nonconstant.

Gauss-Lucas’ theorem states that the zeros of the derivative of a nonconstant univariate polynomial lie in the convex hull of the zeros of the polynomial. If $Q_w'(\zeta) = 0$ for some $\zeta \in H_+$ then by Gauss-Lucas theorem $Q_w$ has a zero $\zeta' \in H_+$, in which case $P(\zeta', w_1, \ldots, w_{m-1}) = Q_w(\zeta') = 0$ which contradicts stability of $P$.

The limit argument is due to Wagner [13].

(vi): Trivial for $d_1 = 0$. Otherwise, this again follows by considering Theorem 2.4 and the symbol of the transformation. In this case,

$$G_T(x, w) = ((\alpha + \beta t)(x_1 + w_1) - d_1) \cdot (x_1 + w_1)^{d_1-1} \prod_{j=2}^m (x_j + w_j)^{d_j}.$$  

7
All factors but the first are present in the stable polynomial $\prod_{j \in [m]} (x_j + w_j)^{d_j}$. The first factor, $(\alpha + \beta t)(x_1 + w_1) - d_1$, is zero only if $\arg(\alpha + \beta t)(x_1 + w_1) = \arg d_1 = 0 \mod 2\pi$. As for (iii), if $(x, w, t) \in H_{+}^{2m+1}$ then $\text{Im}(\alpha + \beta t) \geq 0$ and $\text{Im}(x_1 + w_1) > 0$, in which case their product will never be a positive real number. Hence all factors are stable and so is $G_T$. 

### 2.3 Elementary symmetric polynomials, operators and an inequality

Let’s introduce the elementary symmetric polynomials $e_k(x) \in \mathbb{R}[x_1, \ldots, x_m]$, 

$$e_k(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq m} x_{i_1} \cdots x_{i_k}. \quad \text{(8)}$$

For example, $e_2(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1$. $e_0(x)$ is defined to be 1.

**Proposition 2.6.** The elementary symmetric polynomials are real stable.

**Proof.** Consider the polynomial $P(t, x_1, \ldots, x_m) = \prod_{j \in [m]} (t + x_j) \in \mathbb{R}[t, x_1, \ldots, x_m]$. It is a product polynomial, and hence real stable. Furthermore, expanding the product, 

$$P(t, x_1, \ldots, x_m) = \sum_{k=0}^{m} t^{m-k} e_k(x). \quad \text{(9)}$$

Applying Lemma 2.5 (v) $m - k$ times and (i) to set $t = 0$, we find that 

$$\frac{1}{(m-k)!} \frac{\partial^{m-k} P}{\partial t^{m-k}} \bigg|_{t=0} = e_k(x) \quad \text{(10)}$$

is stable. 

From the proof we can coincidentally deduce Vieta’s formula - for a polynomial $p(t) = \sum_{k=0}^{m} a_k t^k$ with zeros $r_1, \ldots, r_m$, $e_k(r) = (-1)^k \frac{a_k}{a_m}$.

Inspired by the elementary symmetric polynomials, there is another stability-preserving operator we will be interested in: the corresponding elementary symmetric differential operators $e_k(\partial) : \mathbb{R}[x_1, \ldots, x_m] \to \mathbb{R}[x_1, \ldots, x_m]$ are defined as

$$e_k(\partial) = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}}. \quad \text{(11)}$$

**Proposition 2.7.** The elementary symmetric differential operators preserve stability.

8
Proof. Given a real stable polynomial \( P \in \mathbb{R}[x_1, \ldots, x_m] \), applying \( (\alpha + \beta t - \partial / \partial x_j) \) for \( j \in [m] \) with \( \alpha = 0, \beta = 1 \) gives the (by Lemma 2.5 (vi) stable) polynomial
\[
Q(x_1, \ldots, x_m, t) = \prod_{j \in [m]} \left( t - \frac{\partial}{\partial x_j} \right) P = \sum_{k=0}^{m} t^{m-k} (-1)^k e_k(\partial) P. \tag{12}
\]
Lemma 2.5 allows us to extract stability of individual \( t \)-coefficients,
\[
\left. \frac{(-1)^k}{(m-k)!} \frac{\partial^{m-k} Q}{\partial t^{m-k}} \right|_{t=0} = e_k(\partial) P. \tag{13}
\]

An alternate proof of stability of the \( e_k(x) \) emerges: Let \( P = \prod_{j \in [m]} x_j \), a product polynomial and hence stable. Applying the stability preserving \( e_{m-k}(\partial) \),
\[
e_{m-k}(\partial) P = e_k(x). \tag{14}
\]

One of the important properties of stability is that it implies several inequalities on the coefficients. This is the main concern of this thesis. As a first example, we have Newton’s inequalities for the \( e_k \):

**Theorem 2.8** (Newton’s inequalities). Let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). It holds that, for \( 1 \leq k \leq n - 1 \),
\[
\frac{e_k(x)^2}{\binom{n}{k}^2} \geq \frac{e_{k-1}(x)}{\binom{n}{k-1}} \frac{e_{k+1}(x)}{\binom{n}{k+1}}. \tag{15}
\]

The inequality is immediately implied by the following lemma for stable univariate polynomials in general, since the \( e_k \) are coefficients of the polynomial \( p(t) = \prod_{i=1}^{m} (t + x_i) \).

**Lemma 2.9.** For a real-rooted polynomial \( p = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x] \), it holds for \( 1 \leq k \leq n - 1 \) that
\[
\frac{a_k^2}{\binom{n}{k}^2} \geq \frac{a_{k-1} n}{(k-1)} \frac{a_{k+1} n}{(k+1)}. \tag{16}
\]

Proof. Applying Lemma 2.5 (iii) and (v), the polynomial
\[
\frac{d^{n-k+1}}{dt^{n-k+1}} \left[ t^{n-k+1} \frac{d^{k-1} p}{dt^{k-1}} \left( \frac{1}{t} \right) \right] = n! \left( \frac{a_{i-1}}{2(i-1)} t^2 + \frac{a_i}{i} t + \frac{a_{i+1}}{2(i+1)} \right) \tag{17}
\]
is real-rooted. Hence the discriminant is nonnegative, recovering precisely our inequality.

As a parenthesis, most of our polynomials will have non-negative coefficients. This has an important consequence for real-rooted univariate polynomials.
Proposition 2.10. Let $P = \sum_{k=0}^{d} a_k x^k \in \mathbb{R}[x]$ be a real-rooted polynomial. The coefficients of $P$ all have the same sign if and only if the roots of $P$ are non-positive.

Proof. If $P = 0$, then there are no roots and the statement holds vacuously.

Let all of the $a_k$ be non-negative, not all zero. For any positive $x$, $P(x) = \sum_{k=0}^{d} a_k x^k$ is a sum of non-negative numbers not all zero and hence $P(x) > 0 - P$ has no positive roots.

If the $a_k$ are all non-positive then the same reasoning applies.

Now assume $P$ has non-positive roots $-\alpha_1, \ldots, -\alpha_d \in \mathbb{R}$. Then, for some constant $C \in \mathbb{R}$, $P = C \cdot \prod_{j=1}^{d} (x + \alpha_j) = C \cdot \sum_{k=0}^{d} x^{d-k} e_k(\alpha)$. Since $\alpha_j \geq 0$ for each $j \in [d]$, it follows that $e_k(\alpha) \geq 0$ and hence all of the coefficients have the same sign.

2.4 Homogeneity and double stochasticity

There are two other kinds of polynomials we take an interest in, homogeneous and doubly stochastic, which we will introduce here. Double stochasticity can be seen as a generalization of double stochasticity as we know it for matrices: there are two other kinds of polynomials we take an interest in, homogeneous and doubly stochastic.

A polynomial $P \in \mathbb{R}[x_1, \ldots, x_m]$ can be written as $P = \sum_{\alpha \in \mathbb{N}^m} a(\alpha) x^\alpha$, where $a : \mathbb{N}^m \to \mathbb{R}$ has finite support and $x^\alpha$ is taken to be $\prod_{j \in [m]} x_j^{\alpha_j}$.

For a homogeneous polynomial $P$ of degree $d$, $a(\alpha)$ is nonzero only if $|\alpha| = d$, i.e. if each monomial $x^\alpha$ with nonzero coefficient has degree $d$. We write $P = \sum_{|\alpha| = d} a(\alpha) x^\alpha$.

Determinantal polynomials $P = \det \left( \sum_{j \in [m]} x_j A_j \right)$ are homogeneous when $A_0 = 0$, as are product polynomials $P_A = \prod_{i \in [d]} \sum_{j \in [m]} a_{ij} x_j$. Each monomial term in these polynomials is a product of degree-one monomials, $m$ or $d$ respectively.

A homogeneous polynomial $P \in \mathbb{R}[x_1, \ldots, x_m]$ is doubly stochastic if $\frac{\partial P(1)}{\partial x_j} = \frac{d}{m}$ for all $j \in [m]$, where $1 = (1, \ldots, 1) \in \mathbb{R}^m$ is the all-ones vector. This can be viewed as a generalization of double stochasticity as we know it for matrices:

Recall that stochastic matrices $A \in \mathbb{R}^{d \times m}$ have non-negative entries with row sums $\sum_{i \in [m]} a_{ij} = 1$ (or $A1 = 1$). Such a matrix is doubly stochastic if, in addition, it has column sums $\sum_{i \in [d]} a_{ij} = \frac{d}{m} (1A = \frac{d}{m} 1^\top)$.

For the product polynomial of a doubly stochastic matrix $A$,

$$\frac{\partial P_A(1)}{\partial x_k} = \frac{\partial}{\partial x_k} \prod_{i \in [d]} \sum_{j \in [m]} a_{ij} x_j \bigg|_{x=1} = \sum_{l \in [d]} a_{lk} \prod_{i \in [d] \setminus \{l\}} \sum_{j \in [m]} a_{ij} \cdot 1 = \sum_{l \in [d]} a_{lk} = \frac{d}{m},$$

(18)

hence $P_A$ is doubly stochastic.
We will frequently use the following property of homogeneous polynomials in general.

**Proposition 2.11.** Let $P \in \mathbb{R}[x_1, \ldots, x_m]$ be homogeneous of degree $d$ and $D$ the differential operator $DP = \sum_{j \in [m]} \frac{\partial P}{\partial x_j}$. Then, for positive integers $k$,

$$D^k P(\mathbf{1}) = (d)_k P(\mathbf{1}),$$

(19)

where $(n)_k$ is the Pochhammer symbol defined for non-negative integers $n, k$ as

$$(n)_k = \prod_{j=n-k+1}^{n} j = n \cdot (n-1) \cdot \ldots \cdot (n-k+1).$$

(20)

The empty product is 1 by convention.

Note that for $k = 1$ and $P$ doubly stochastic, Proposition 2.11 implies $P(\mathbf{1}) = 1$.

**Proof.** By induction on $k$.

The base case $k = 1$ immediately follows from differentiating the identity $P(tx_1, \ldots, tx_m) = t^d P(x_1, \ldots, x_m)$ with respect to $t$ and setting $t = 1$.

For $k > 1$, let $Q = DP$, a homogeneous of degree $d - 1$. By applying the induction hypothesis and the base case,

$$D^k P(\mathbf{1}) = D^{k-1} Q(\mathbf{1}) = (d - 1)_{k-1} Q(\mathbf{1}) = (d - 1)_{k-1} \cdot d \cdot P(\mathbf{1}) = (d)_k P(\mathbf{1}).$$

(21)

\[ \square \]

### 2.5 A probabilistic perspective

Let $P \in \mathbb{R}[x_1, \ldots, x_m]$ be a polynomial with non-negative coefficients and $P(\mathbf{1}) = 1$. There's a probabilistic interpretation of $P$ as the multivariate probability generating functions (p.g.f.) of certain probability distributions, and establishing inequalities on the coefficients of $P$ will hence give information about the underlying probability distributions.

Given random variables $X_1, \ldots, X_m$ on $\mathbb{N}$ with finite support, define for a multi-index $\alpha$,

$$a(\alpha) = \Pr [X_j = \alpha_j : j \in [m]].$$

(22)

The polynomial $P = \sum_{\alpha} a(\alpha)x^\alpha$ is the p.g.f. of the multivariate random variable $X = (X_1, \ldots, X_j)$. Conversely, any polynomial $Q \in \mathbb{R}[x_1, \ldots, x_m]$ with non-negative entries and $Q(\mathbf{1}) = 1$ defines a multivariate random variable of which $Q$ is the PGF.

These distributions can be thought of as an urn model with $m$ labeled urns and $X_j$ identical balls in the $j$th urn.
Considering arbitrary homogeneous polynomials \( P \) of degree \( d \) for which the coefficients are non-negative and \( P(1) \), it imposes the restriction \( \sum_{j \in [d]} X_j = d \). They are the p.g.f.s of probability distributions on configurations on \( d \) balls in \( m \) labeled urns.

If we generate a configuration on \( d \) balls by starting with empty urns and placing the \( i^{th} \) ball in the \( j^{th} \) urn with probability \( a_{ij} \), with \( \sum_{j \in [m]} a_{ij} = 1 \), then the p.g.f. is the homogeneous product polynomial of the stochastic \( d \times m \) matrix \( A \) with the \( a_{ij} \) as entries. \( X \) can in this case be decomposed into the sum of \( d \) multinoulli-distributed variables \( I_i \) on the unit vectors, \( i \in [d] \), each nonzero in the \( i^{th} \) entry with probability \( a_{ij} \) and representing the position of the \( i^{th} \) ball.

\[
\begin{align*}
\frac{d}{m} &= \frac{\partial P(1)}{\partial x_j} \\
&= \sum_{|\alpha|=d} a(\alpha)a_{j\cdot}1 = \sum_{|\alpha|=d} \alpha_j \cdot \Pr[X = \alpha] = \mathbb{E}[X_j].
\end{align*}
\] (23)

The expected number of balls in each urn is \( \frac{d}{m} \), independent of \( j \).

By Proposition \ref{prop:2.11} with \( k = 1 \), a doubly stochastic polynomial has \( P(1) = 1 \) and can therefore always be taken to be a p.g.f. Furthermore, the restriction on derivatives in \( \mathbb{1} \) imply something about the urn configurations:

\[
e_k(\partial)P(\mathbb{1}) = \mathbb{E}[e_k(X)].
\] (24)
Restricting the values $e_k(\partial)P(1)$ hence restricts the elementary symmetric moments of the $X$. This is precisely what happens in Leonid Gurvits’ polynomial version of van der Waerden’s permanent conjecture, given in the next section as Theorem 3.1. If the p.g.f. is doubly stochastic, i.e. if $E[X] = 1$ and $\sum_j X_j = m$, then stability of the p.g.f. implies

$$e_m(\partial)P(1) = E [X_1 \cdot X_2 \cdots X_m] \geq \frac{m!}{m^m}. \quad (25)$$
3 Gurvits’ theorem

Gurvits’ theorem is the main inspiration for this thesis. It relates the coefficient of $\prod_{j \in [m]} x_j$ of a stable polynomial to a measure called its capacity.

The theorem is introduced in Section 3.1 together with a brief extension of its applicability. Section 3.2 contains a proof, followed in Section 3.3 by a polynomial for which the inequality in Gurvits’ theorem is tight. Finally, in Section 3.4, a generalization of Gurvits’ theorem, for doubly stochastic polynomials, is given with a proof due to Brändén.

3.1 Statement

The capacity of a homogeneous polynomial $P \in \mathbb{R}[x_1, \ldots, x_m]$ with degree $m$ is

$$\text{Cap}(P) = \inf_{x \in \mathbb{R}^m_+} \frac{P(x_1, \ldots, x_m)}{x_1 \cdots x_m}. \quad (26)$$

Define $G(0) = G(1) = 1, G(k) = \left(1 - \frac{1}{k}\right)^{k-1}$ for positive integers $k \geq 2$.

**Theorem 3.1** (Gurvits’ theorem). Let $P \in \mathbb{R}[x_1, \ldots, x_m]$ be a stable, homogeneous polynomial with nonnegative coefficients, of total degree $m$, and of degree $d_j$ in $x_j$ for each $j \in [m]$. Let $f_j = \min\{j, d_j\}$. Then

$$\text{Cap}(P) \geq \frac{\partial^m}{\partial x_1 \cdots \partial x_m} P(0, \ldots, 0) \geq \text{Cap}(P) \prod_{j=1}^m G(f_j) \geq \text{Cap}(P) \frac{m!}{m^m}. \quad (27)$$

The evaluation at 0 is not particularly special. If we drop the first inequality then we can obtain the theorem for any point $y \in \mathbb{R}^m_0$.

**Corollary 3.2.** Let $P \in \mathbb{R}[x_1, \ldots, x_m]$ be as above and fix $y \in \mathbb{R}^m_0$. Then

$$\frac{\partial^m}{\partial x_1 \cdots \partial x_m} P(y) \geq \text{Cap}(P) \prod_{j=1}^m G(f_j) \geq \text{Cap}(P) \frac{m!}{m^m}. \quad (28)$$

**Proof.** Let $Q(x) = P(x + y)$. The definition of the $f_j$ gives the same numbers when applied to $P$ and $Q$. We have, applying Theorem 3.1

$$\frac{\partial^m}{\partial x_1 \cdots \partial x_m} P(y) = \frac{\partial^m}{\partial x_1 \cdots \partial x_m} Q(0, \ldots, 0), \quad (29)$$

$$\geq \text{Cap}(Q) \prod_{j=1}^m G(f_j). \quad (30)$$
Cap(Q) is in turn an upper bound of Cap(P):

\[
\begin{align*}
\text{Cap}(Q) &= \inf_{x \in \mathbb{R}^m_+} \frac{Q(x)}{x_1 \cdots x_m}, \\
&= \inf_{x \in \mathbb{R}^m_+} \frac{P(x + y)}{x_1 \cdots x_m}, \\
&\geq \inf_{x \in \mathbb{R}^m_+} \frac{P(x + y)}{(x_1 + y_1) \cdots (x_m + y_m)}, \\
&\geq \inf_{x \in \mathbb{R}^m_+} \frac{P(x)}{(x_1) \cdots (x_m)}, \\
&= \text{Cap}(P).
\end{align*}
\]

The lemma follows directly.

\[\square\]

### 3.2 Proof of Gurvits’ Theorem

The proof presented in this section can be found in Brändén’s unpublished lecture notes [12]. It is basically Gurvits’ original proof found as Theorem 2.4 in [2]. The proof is inductive and relates the capacity of a polynomial to its derivative at zero. By repeatedly applying this, differentiating once in each variable \(x_i\), we are left with the constant polynomial \(\frac{\partial^m}{\partial x_1 \cdots \partial x_m} P(0, \ldots, 0)\), precisely the multilinear term we are after.

We will need two simple lemmas.

**Lemma 3.3.** If \(Q \in \mathbb{R}[x]\) is a real-rooted degree-\(d\) polynomial with non-negative coefficients, then

\[
Q'(0) \geq G(d) \cdot \text{Cap}(Q).
\]  

**Proof.** As the coefficients are non-negative, Proposition [2.10] applies and hence \(Q\) has no positive zeros. Let \(Q = c \prod_{i=1}^{d} (x + \alpha_i)\) for some \(c, \alpha_i \in \mathbb{R}_{\geq 0}\).

We may assume that \(Q(0) \neq 0\), i.e. \(\alpha_i > 0\) for all \(i\): Otherwise we can consider the sequence of polynomials \(Q_k(x) = Q(x + \frac{1}{k})\), satisfying the conditions of the lemma and hence taking the limits of the inequality proves the lemma for \(Q\).

By the AM-GM inequality, with \(w_i = \frac{1}{\alpha_i}, \bar{w} = \frac{1}{d} \sum_{i=1}^{d} w_i\) and \(c' = Q(0)\),

\[
\frac{Q(x)}{x} = \frac{1}{c'} \prod_{i=1}^{d} \left(1 + x w_i\right) \leq c' \frac{(1 + x \bar{w})^d}{x}.
\]

The function \(f(x) = \frac{(1+x\bar{w})^d}{x}\) has a global minimum at \(x^* = \frac{1}{(d-1)\bar{w}}\). \(f(x^*) = \frac{d\bar{w}}{G(d)}\).

Noting that \(Q'(0) = d\bar{w}\), we have that \(\frac{Q(x)}{x} G(d) \leq Q'(0)\). Taking the infimum over \(x\) finishes the proof. \(\square\)
The next lemma is the workhorse of the proof, driving the induction.

**Lemma 3.4.** If \( P \in \mathbb{R}[x_1, \ldots, x_m] \) is a stable polynomial with non-negative coefficients, \( Q = \frac{\partial P}{\partial x_i} \bigg|_{x_i=0} \in \mathbb{R}[\{x_1, \ldots, x_m\}\setminus\{x_i\}] \) and \( k = \deg_{x_i}(P) \), then

\[
\text{Cap}(Q) \geq G(k) \cdot \text{Cap}(P).
\]

**Proof.** Treating \( P \) as a univariate polynomial in \( m \), we apply Lemma 3.3 and find

\[
\frac{Q(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)}{x_1 \cdots x_{i-1} x_{i+1} \cdots x_m} \geq G(k) \cdot \inf_{x_i > 0} \frac{P(x_1, \ldots, x_m)}{x_1, \ldots, x_m}.
\]

Taking the infimum over positive reals on the remaining variables proves the lemma.

**Proof of Gurvits’ Theorem.** Define a sequence of polynomials \( Q_k = \frac{\partial^k P}{\partial x_1 \cdots \partial x_k} \bigg|_{x_1=\cdots=x_k=0} \in \mathbb{R}[x_{k+1}, \ldots, x_m] \) for \( k = 0, \ldots, m \), where \( Q_0 = P \) and \( Q_m = \frac{\partial^m P(0, \ldots, 0)}{\partial x_1 \cdots \partial x_m} \).

The first inequality \( \text{Cap}(P) \geq Q_m \) is trivial, as \( Q_m \) is a constant contribution to \( \frac{P(x_1, \ldots, x_m)}{x_1 \cdots x_m} \) in the infimum.

The degree of \( x_k \) in \( Q_{k-1} \) is at most \( f_i = \min\{i, d_i\} \) and, since \( G \) is non-increasing, \( G(\deg_{x_i} Q_{i-1}) \geq G(e_i) \). Lemma 3.3 applies to each \( Q_{i-1} \) and its derivative at \( x_i = 0, Q_i \) for \( i \in [m] \). By induction, we get a sequence

\[
Q_m = \text{Cap}(Q_m) \geq G(e_m) \text{Cap}(Q_{m-1}) \geq \cdots \geq \prod_{j=1}^{m} G(f_j) \text{Cap}(P).
\]

Noting that \( \prod_{j=1}^{m} G(f_j) \geq \prod_{j=1}^{m} G(j) = \frac{m!}{m^m} \), we are done.

**3.3 Tightness**

Let \( J_{d,m} \) be the \( d \times m \)-matrix consisting of all ones. We will show that Gurvits’ theorem is in general tight but has room for improvement.

The upper bound on the multilinear coefficient is trivially tight for the product polynomial of the identity matrix. The other bound is tight for the product polynomial of \( \frac{1}{m} J_{m,m} \).

The capacity is easily handled since \( P_{J_{d,m}} \) has capacity one.

**Proposition 3.5** ([12]). A homogeneous polynomial \( P \in \mathbb{R}[x_1, \ldots, x_m] \) with non-negative weights is doubly stochastic if and only if \( \text{Cap}(P) = P(1) = 1 \).
Proof. Let \( P = \sum_{\alpha} a(\alpha) x^{\alpha} \) be doubly stochastic. Then \( \sum_{\alpha} a(\alpha) = 1 \) and the coefficients act as weights in the weighted AM-GM inequality for \( x \in \mathbb{R}^m_+ \),

\[
P(x) = \sum_{\alpha} a(\alpha) x^{\alpha} \geq \prod_{\alpha} x^{\alpha a(\alpha)} = x^{\sum_{\alpha} \alpha a(\alpha)} = x^{d_1 m}.
\] (41)

We find that \( \text{Cap}(P) \geq \inf_{x_1, \ldots, x_m \geq 0, x_1 + \cdots + x_m = 1} x^{d_1 m} \) and the coefficients act as weights in the weighted AM-GM inequality for \( x \in \mathbb{R}^m_+ \),

\[
\sum_{\alpha} a(\alpha) x^{\alpha} \geq \prod_{\alpha} x^{\alpha a(\alpha)} = x^{\sum_{\alpha} \alpha a(\alpha)} = x^{d_1 m}.
\]

Conversely if \( \text{Cap}(P) = P(1) = 1 \) then the function \( f : (-1, 1) \to \mathbb{R} \) defined by \( f(t) = P(1 + t\tilde{e}_i - t\tilde{e}_j)/(1 + t^2) \), where \( \tilde{e}_i \) is the \( i \)th standard basis vector of \( \mathbb{R}^m \), has a minimum in \( t = 0 \), i.e.

\[
\frac{df(0)}{dt} = \frac{\partial P(1)}{\partial x_i} - \frac{\partial P(1)}{\partial x_j} = 0.
\] (42)

Together with Proposition 2.11 with \( k = 1 \), it follows that \( \frac{\partial P(1)}{\partial x_i} = \frac{d}{m} \) and hence \( P \) is doubly stochastic.

The multilinear coefficient of \( P_{j, m} \) is \( m!/m^m \), and hence it is tight in Gurvits’ theorem.

\[
P_{j, m} = \left(\sum_{i=1}^{m} x_i + \cdots + x_m \right)^m,
\]

and by expanding this, terms contributing to the multilinear factor chooses precisely one factor for each variable. As we will see in Chapter 5, it holds in general that the multilinear coefficient of \( P_A \) is the permanent of the matrix \( A \).

Gurvits’ theorem gives a stronger bound on the coefficient of \( \prod_{j \in [m]} x_j \) if the polynomial is bounded in individual degrees:

Define a polynomial \( P \in \mathbb{R}[x_1, \ldots, x_m] \) to be individually degree-bounded by \( d' \) if \( \deg_{x_j} P \leq d' \) for all \( j \in [m] \). This implies \( f_j \leq d' \) for \( j \geq d' \) in Gurvits’ theorem. For example if \( P \in \mathbb{R}[x_1, x_2, x_3, x_4] \) is individually degree-bounded by 3 then Gurvits’ theorem gives us the bound \( e_4(\theta) P(1) \geq \frac{8}{m} \text{Cap}(P) \). However, the author has been unable to find polynomials for which this value lies below \( \frac{1}{2} \).

In the case \( d' = 2 \), however, the inequality is still tight:

The polynomial \( P_0 = 2^{-m} \prod_{j \in [m]} (x_j + x_{j+1}) \), indices taken cyclically, has capacity one (doubly stochastic) and, expanding the product, only contributes with two terms \( 2^{-m} \) to the multilinear coefficient (differentiating in one variable in a factor fixes the remaining ones). Hence its multilinear coefficient is \( 2^{-m+1} \), which is precisely the right-hand side of Gurvits’ inequality with \( f_j = \min(j, 2) \).
3.4 An extension for doubly stochastic polynomials

Gurvits’ theorem is a situation where double stochasticity proves to be a well-behaved normalization that reduces the inequality to an absolute bound by Proposition 3.5.

**Theorem 3.6.** Let $P \in \mathbb{R}[x_1, \ldots, x_m]$ be a stable, homogeneous, doubly stochastic polynomial of degree $d$ with nonnegative coefficients. Then

$$e_k(\partial)P(1) \geq k! \binom{m}{k} \binom{d}{k} m^{-k}. \quad (43)$$

Equality is achieved if and only if $P = P_{J_{d,m}}$.

This theorem lies close to the Friedland-Tverberg inequality, Theorem 3.1 in [14], and they agree in the case $k = d$.

The main idea behind the proof is to construct a polynomial $Q$ of degree $m$ from $P$ and then apply Corollary 3.2.

**Proof.** Recall the operator $D = \left( \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_m} \right)$.

First, let $d \leq m$ and $Q = \left( \frac{x_1 + \cdots + x_m}{m} \right)^{m-d+k} \frac{D^kP}{(d)_k}$ for $0 \leq k \leq d$. The polynomial $Q$ is doubly stochastic, by evaluating and applying Proposition 2.11.

\[
\frac{\partial Q}{\partial x_i} = \frac{m - d + k}{m} \left( \frac{x_1 + \cdots + x_m}{m} \right)^{m-d+k-1} \frac{D^kP}{(d)_k} \left( \frac{x_1 + \cdots + x_m}{m} \right) \quad (44)
\]

\[
\frac{\partial Q(1)}{\partial x_i} = \frac{m - d + k}{m} + \frac{D^k\frac{\partial P(1)}{\partial x_i}}{(d)_k}, \quad (45)
\]

\[
= \frac{m - d + k}{m} + \frac{(d-1)_k}{(d)_k} \cdot \frac{d}{m}, \quad (46)
\]

\[
= \frac{d}{m}. \quad (47)
\]

As $D$ and multiplication is stability-preserving, $Q$ is stable and we find that $Q$ satisfies the conditions of Corollary 3.2 with $\text{Cap}(Q) = 1$ by Proposition 3.5. We have

\[
\frac{\partial^m Q(1)}{\partial x_1 \cdots \partial x_m} \geq \frac{m!}{m^m}, \quad (48)
\]

with equality if $Q = P \frac{1}{m^m} J_{m,m}$, i.e. if $P = \left( \frac{x_1 + \cdots + x_m}{m} \right)^d$. 

18
It remains to express the linear coefficient of $Q$ in terms of $P$.

\[
\frac{\partial^m Q(\mathbf{1})}{\partial x_1 \cdots \partial x_m} = \sum_{\alpha \subseteq [m], |\alpha| = d-k} \frac{\partial^{m-d+k}}{\partial x_{[m]\setminus \alpha}} \left( \frac{x_1 + \cdots + x_m}{m} \right)^{m-d+k} \cdot \frac{1}{(d)_k} \frac{\partial^{d-k} D^k P}{\partial x_{\alpha}} \Big|_1. \tag{50}
\]

One of the two factors will disappear unless it holds that $m - |\alpha| \leq m - d + k$ and $|\alpha| + k \leq d$ respectively, i.e. $|\alpha| = d - k$.

\[
\frac{\partial^m Q(\mathbf{1})}{\partial x_1 \cdots \partial x_m} = \sum_{\alpha \subseteq [m], |\alpha| = d-k} \frac{\partial^{m-d+k}}{\partial x_{[m]\setminus \alpha}} \left( \frac{x_1 + \cdots + x_m}{m} \right)^{m-d+k} \cdot \frac{1}{(d)_k} \frac{\partial^{d-k} D^k P}{\partial x_{\alpha}} \Big|_1, \tag{51}
\]

\[
= \frac{(m - d + k)!}{m^{m-d+k}(d)_k} D^k e_{d-k}(\partial) P(\mathbf{1}), \tag{52}
\]

\[
= \frac{(m - d + k)!}{m^{m-d+k}(d)_k} e_{d-k}(\partial) P(\mathbf{1}), \tag{53}
\]

where we applied Proposition 2.11 for the final equality. We conclude, substituting $k \leftarrow d - k$,

\[
\frac{(m - k)!}{m^{m-k}(d)_k} e_k(\partial) P(\mathbf{1}) \geq \frac{m!}{m^m}, \tag{54}
\]

\[
e_k(\partial) P(\mathbf{1}) \geq k \binom{m}{k} \binom{d}{k} m^{-k}. \tag{55}
\]

Equality follows from equality in (49), i.e. if $P = P_{d,m}$.

For the case $d > m$, let $Q = \left( \frac{x_1 + \cdots + x_m}{m} \right)^{m-d+k} D^k P \Big|_{(d)_k}$ for $d - m \leq k \leq d$. The proof proceeds identically.

We refrain from proving the only if-part of equality for $P_{d,m}$. It can be done using the theory of hyperbolic polynomials by following the proof of Theorem 11.9 in [12].
4 Mixed characteristic polynomials

The seminal articles on infinite families of bipartite Ramanujan graphs [5] and the Kadison-Singer conjecture [6] of Marcus, Spielman and Srivastava introduce the \textit{mixed characteristic polynomial} which we abbreviate \textit{MCP}. We give a definition and explore some properties in the first section of this chapter. In Section 4.2 we compute the MCP of a combinatorially interesting polynomial. The remaining sections deal with a linear operator \( \mathbb{R}[x_1, \ldots, x_m] \to \mathbb{R}[x_1, \ldots, x_m] \) and its properties (Section 4.3) and how it transforms the underlying MCP (Sections 4.4, 4.5).

4.1 Definition and properties

The \textit{mixed characteristic polynomial} \( \chi_t[P] \) of a stable, homogeneous polynomial \( P \in \mathbb{R}[x_1, \ldots, x_m] \) is defined as

\[
\chi_t[P] = \prod_{j=1}^{m} \left( 1 - \frac{\partial}{\partial x_j} \right) P \bigg|_{x_1 = \cdots = x_m}.
\]  

(56)

Recognizing a parallel to the discussion on elementary symmetric polynomials and in particular (9), the MCP expands in terms of the operators \( e_k(\partial) \).

\[
\chi_t[P] = \sum_{k=0}^{m} (-1)^k e_k(\partial) P(1) t^{d-k},
\]  

(57)

where \( d \) is the degree of \( P \). Note that if \( k > d \) then \( e_k(\partial)P = 0 \) so it is indeed a polynomial.

We can immediately draw conclusions about the roots of \( \chi_t[P] \).

**Proposition 4.1.** The MCP is real-rooted with non-negative roots.

\textit{Proof.} By Proposition 2.5 the operator \( \left( 1 - \frac{\partial}{\partial x_j} \right) \) is a stability preserver, as is evaluating in \( x_1 = \cdots = x_m \). Hence the MCP is real stable and thus real-rooted.

Furthermore, the coefficients of the polynomial \( (-1)^k e_k(\partial)P(1) t^{d-k} \) are all non-negative. Hence it has non-positive zeros by Proposition 2.10 and so \( \chi_t[P] \) has non-negative roots.

\( \square \)

In this setting, Theorem 3.1 bounds the constant term of \( \chi_t[P] \) for \( m \)-degree polynomials in \( m \) variables and Brändén’s Gurvits-like inequality in Theorem 3.6 bounds all of the coefficients of arbitrary MCPs (of stable, doubly stochastic polynomials) with degree \( d \leq m \).
A question raised by Marcus, Spielman and Srivastava is what can be said of the roots of $\chi_t[P]$. Bounding the roots is an integral part of the proof of the Kadison-Singer conjecture, and finding alternate proofs of their bounds would circumvent the cumbersome barrier argument in Section 5 of [6].

Another important question they raise [6] is if their bound can be improved. They conjecture that the largest root of $\chi_t[P]$ is maximized by the polynomial $P_{J_d,m}$ introduced in Section 3.3, a scaled associated Laguerre polynomial as we will see in Section 4.5.

### 4.2 A combinatorial MCP example

To illustrate the combinatorial aspects of the MCP, we give a computation for the polynomial $P_0 = \prod_{j=1}^{m}(x_j + x_{j+1})$ introduced in Section 3.3.

The polynomial $P_0$ is of degree $d = m$ and individual degrees $\deg_{x_j} P_0 = 2$. This is the product polynomial of a matrix with two cyclic all-ones diagonals, which as we will see in Chapter 5 is naturally seen as the bi-adjacency matrix of a cycle on $2m$ vertices.

The map $P \mapsto \chi_P$ is a linear operator for polynomials $P \in \mathbb{R}[x_1, \ldots, x_m]$ so we can treat monomial terms of $P$ in isolation. We have

$$e_k(\partial) [x_1^{\alpha_1} \cdots x_m^{\alpha_m}] (1) = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \alpha_{i_1} \cdots \alpha_{i_k} = e_k(\alpha), \quad (58)$$

and hence

$$\chi_t[x^{\alpha}] = \sum_{k=0}^{m} (-1)^k e_k(\alpha) t^{d-k} = t^{d-m} \prod_{j=1}^{m} (t - \alpha_j). \quad (59)$$

In our case, the degree bounds on $P_0$ imply that we can write $P_0$ as

$$P_0 = \sum_{\alpha \in \{0,1,2\}^m, |\alpha| = m} c_{\alpha} x^{\alpha}. \quad (60)$$

The MCP mapping is oblivious to the order of entries of $\alpha$ so it is sufficient to count the number of 0’s, 1’s and 2’s in $\alpha$. With $s$ 0’s, $u$ 1’s and $v$ 2’s in $\alpha$, we have $|\alpha| = u + 2v = m$ such that $s + u + v = m$. The solution is $(s,u,v) = (s, m-2s, s)$ for $0 \leq s \leq \lfloor \frac{m}{2} \rfloor$.

Applying the expression (58) to $P_0$, we have

$$\chi_t[P_0] = \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} b_s (t - 2)^s(t - 1)^{m-2s} t^s, \quad (61)$$

where $b_s$ is the number of ways to choose $\alpha \in \{0,1,2\}^m$ such that $|\alpha| = m$ and $\alpha_i = \alpha_{i+1} = 2$ never occurs. This is the number of ways of orienting the edges of the cyclic graph $C_m$ such that precisely $s$ vertices have two incoming edges. Such a digraph has
2s acyclic paths and is completely specified by \(2s\) vertices in which to change direction together with an orientation. Hence \(b_s = 2\binom{m}{2s}\).

\[
\chi_t[P_0] = \sum_{s=0}^{\lfloor \frac{m}{2s} \rfloor} 2 \binom{m}{2s} (t-2)^s(t-1)^{m-2s} t^s.
\]

(62)

### 4.3 A resampling operator

One approach to the MCP is to study how it evolves under certain transformations. In this section we introduce an operator with this purpose. We derive it in the probabilistic setting of Section 2.5.

Given an outcome \(X\) from some distribution on urn configurations, with homogeneous, degree \(d\) p.g.f. \(P\), we can uniformly randomly remove one of the balls in the urns to obtain a distribution on \(d-1\) balls. With the vector \(\alpha \in \mathbb{N}^m\) describing the number of balls in each urn, we will remove a ball from the \(j\)th urn with probability \(\alpha_j/d\), and change the vector to \((\alpha_1, \ldots, \alpha_{j-1}, \alpha_j-1, \alpha_{j+1}, \ldots, \alpha_m)\). This operation hence transforms the p.g.f. by

\[
P = \sum_{\alpha} a(\alpha) x^\alpha \mapsto \sum_{\alpha} a(\alpha) \sum_{\text{choose ball from urn } j} \alpha_j x^\alpha / x_j,
\]

(63)

\[
= \sum_{\alpha} a(\alpha) \sum_{j \in [m]} \frac{\partial}{\partial x_j} x^\alpha,
\]

(64)

\[
= \frac{1}{d} DP,
\]

(65)

where we recall the operator \(D = \sum_{j \in [m]} \frac{\partial}{\partial x_j}\). To reinsert this ball into an urn chosen uniformly at random we simply multiply \(P\) by the polynomial \(Q = x_1^{a_1} + \cdots + x_m^{a_m}\).

We define the composed operation as

\[
T : P \mapsto \frac{1}{d} QDP = \frac{1}{dm} \sum_{i \in [m]} x_i \sum_{j \in [m]} \frac{\partial P}{\partial x_j}.
\]

(66)

\(T\) has several promising properties, testifying to the robustness of the polynomial classes in this thesis.

**Proposition 4.2.** The following holds for the operator \(T\).

(i) \(T\) is a endomorphism on the linear space of homogeneous, degree \(d\) polynomials of \(\mathbb{R}[x_1, \ldots, x_m]\);

(ii) \(T\) preserves stability;

(iii) \(T\) maps p.g.f.s onto p.g.f.s.
(iv) $T$ preserves double stochasticity.

Proof. (i) is immediately clear and (ii) is due to $e_1(\partial) = D$ being a stability preserver (Proposition 2.7) and stability being closed under multiplication. (iii) is a simple application of Proposition 2.11: $TP(1) = \frac{1}{d} QDP(1) = \frac{1}{d} \cdot 1 \cdot dP(1) = 1$ and coefficients of $TP$ are clearly non-negative if those of $P$ are. (iv): $\partial P_{x_j}$ is a homogeneous $(d-1)$-degree polynomial and hence $D \frac{\partial P(1)}{\partial x_j} = (d-1) \frac{\partial P(1)}{\partial x_j} = \frac{d(d-1)}{m}$ and $DP(1) = d$ (Proposition 2.11). We have

$$
\frac{\partial TP(1)}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{1}{d} QDP(1) = \frac{1}{dm} DP(1) + \frac{1}{d} QD \frac{\partial P(1)}{\partial x_j} = \frac{d}{dm} + \frac{1}{d} \frac{d(d-1)}{m} = \frac{d}{m}. \tag{67}
$$

The matrix $\frac{1}{m} J_{d,m}$ from Section 3.3 occurs in several contexts and this is another one. In the urn model of Section 2.5, $J_{d,m}$ represents distributing each of $d$ balls uniformly randomly in $m$ urns. Its product polynomial is the limit distribution when iterating $T$.

Theorem 4.3. For a homogeneous $d$-degree polynomial $P \in \mathbb{R}[x_1, \ldots, x_m]$ with non-negative coefficients and $P(1) = 1$ (i.e. $P$ is a p.g.f.),

$$
\lim_{n \to \infty} T^n P = P_{J_{d,m}}. \tag{68}
$$

Given a p.g.f. $P \in \mathbb{R}[x_1, \ldots, x_m]$, the sequence of functions $\{T^n P\}_n$ represents iterating the action of randomly choosing a ball and reinserting it, an iterated resampling.

The probability of having chosen each ball at least once approaches 1. Informally, the probability of not picking the $m^{th}$ ball after $n$ iterations is $\left(\frac{m-1}{m}\right)^n$, approaching 0 as $n \to \infty$. When each ball has been chosen its position will be uniformly randomly placed, independently of where it lay previously or the other balls, and hence the entire distribution has the p.g.f. $P_{J_{d,m}}$.

We give two proofs. One is based on Markov theory that makes use of the probabilistic intuition and the other proof is a technical one based on expanding $T^n P$ into a finite combination of polynomials where the coefficient of $P_{J_{d,m}}$ dominates the others in the limit $n \to \infty$.

Proof 1: Markov theory. We can view $T$ as the transition function of a Markov chain, taking configurations of $d$ balls in $m$ urns as its finite state space. Consider arbitrary configurations $\alpha, \beta \in \mathbb{N}^m$, $|\alpha| = |\beta| = d$. Starting from state $\alpha$, after $d$ resamplings, the probability of having chosen each ball precisely once is non-zero and the probability of subsequently having placed them in the configuration $\beta$ is non-zero, hence it is possible to reach any state $\beta$ from any state $\alpha$ in $d$ or fewer transitions. Hence our Markov chain is ergodic and has a unique limit distribution, satisfying $TP = P$, which is easily verified to be $P_{J_{d,m}}$. \qed
It is useful to give a more technical proof that reveals more about the behaviour of the convergence. We will need two lemmas about the Stirling numbers of the second kind.

**Lemma 4.4.** Let \( P \in \mathbb{R}[x_1, \ldots, x_m] \) be homogeneous of degree \( d \). It holds that

\[
T^n P = \frac{1}{d^n} \sum_{k=0}^{d} S(n, k) Q^k D^k P, \tag{69}
\]

where \( S(n, k) \) are the Stirling numbers of the second kind.

**Proof.** We prove, by induction on \( n \), that \( T^n P = \frac{1}{d^n} \sum_{k=0}^{d} a(n, k) Q^k D^k P \) for coefficients \( a(n, k) = S(n, k) \), \((n, k) \in \mathbb{N} \times \{0, \ldots, d\}\). The base case \( n = 0 \) is simple, \( a(0, 0) = 1 \) and \( a(0, k) = 0 \) for \( k > 0 \).

We have

\[
T^{n+1} P = TT^n P, \tag{70}
\]

\[
= \frac{1}{d} QD \frac{1}{d^n} \sum_{k=0}^{d} a(n, k) Q^k D^k P, \tag{71}
\]

\[
= \frac{1}{d^{n+1}} \sum_{k=0}^{d} a(n, k) QDQ^k D^k P, \tag{72}
\]

\[
= \frac{1}{d^{n+1}} \sum_{k=0}^{d} a(n, k) \left( kQ^k D^k P + Q^{k+1} D^{k+1} P \right), \tag{73}
\]

\[
= \frac{1}{d^{n+1}} \sum_{k=1}^{d} \left(a(n, k - 1) + k \cdot a(n, k)\right) Q^k D^k P. \tag{74}
\]

Hence the \( a(\cdot, \cdot) \) satisfy \( a(0, 0) = 1 \), \( a(n, 0) = a(0, k) = 0 \) for \( n, k > 0 \) and the recursion

\[
a(n + 1, k) = a(n, k - 1) + k \cdot a(n, k), \tag{75}
\]

defining the Stirling numbers of the second kind (See e.g. [13], page 74).

**Lemma 4.5.** For any positive integer \( k \) it holds that \( S(n, k) \) satisfies

\[
S(n, k) = \frac{k^n}{k!} + f(n, k), \tag{76}
\]

where \( \frac{1}{k!} f(n, k) \to 0 \) as \( n \to \infty \).
Proof. It suffices to show that the difference \( \frac{1}{k^n} S(n,k) - \frac{1}{k!} \) → 0 as \( n \to \infty \). Using the sum given on page 74 of [15],

\[
\left| \frac{1}{k^n} S(n,k) - \frac{1}{k!} \right| = \left| \frac{1}{k^n} \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n - \frac{1}{k!} \right|,
\]

(77)

\[
= \frac{1}{k!} \sum_{j=0}^{k-1} (-1)^{k-j} \binom{k}{j} \left( \frac{j}{k} \right)^n,
\]

(78)

\[
\leq \frac{1}{k!} \sum_{j=0}^{k-1} \binom{k}{j} \left( \frac{j}{k} \right)^n,
\]

(79)

\[
\leq \left( 1 - \frac{1}{k} \right)^n \frac{1}{k!} \sum_{j=0}^{k-1} \binom{k}{j},
\]

(80)

\[
\to 0 \text{ as } n \to \infty.
\]

(81)

The convergence of \( T^n P \) easily follows.

Proof 2: Stirling number expansion. With the expression for \( T \) given in Lemma 4.4,

\[
T^n P = \frac{1}{d^n} \sum_{k=0}^{d} S(n,k) Q^k D^k P,
\]

(82)

\[
= \sum_{k=0}^{d-1} \left[ \frac{k^n}{d^n} + f(n,k) \right] Q^k D^k P + \frac{d^n}{d^n} \left[ \frac{d^n}{d^n} + f(n,d) \right] Q^d D^d P,
\]

(83)

\[
\to 0 + Q^d \frac{1}{d!} D^d P,
\]

(84)

with the convergence implied by Lemma 4.5. As \( P \) is of degree \( d \), any \( d^{th} \) derivative is a constant polynomial and hence by applying Proposition 2.11, \( D^d P = D^d P(1) = d! \).

4.4 Monotonicity of coefficients of certain MCPs

Given the convergence of \( T^n P \) to \( P_{J_{d,m}} \) in Theorem 4.3, it follows that the MCPs also converge, \( \chi_t[T^n P] \to \chi_t[P_{J_{d,m}}] \). Put together with the tightness of \( P_{J_{d,m}} \) in Theorems 3.1 and 3.6, we initially conjectured that the coefficients of \( \chi_t[T^n P] \) approach those of \( \chi_t[P_{J_{d,m}}] \) in a monotone fashion.

This is not true in general, and we will give a counterexample in Chapter 5, but it does hold in special cases. That is the subject of this section.
Proposition 4.6. For a homogeneous polynomial \( P \in \mathbb{R}[x_1, \ldots, x_m] \) with degree \( d \) and non-negative coefficients, and an integer \( k > 0 \),

\[
e_k(\partial)P(1) \geq \frac{(m - k + 1)(d - k + 1)}{km} e_{k-1}(\partial)P(1).
\] (85)

This is immediately implied by expressing the coefficients of \( \chi_t[TP] \) in terms of those of \( \chi_t[P] \):

Lemma 4.7. For a homogeneous polynomial \( P \in \mathbb{R}[x_1, \ldots, x_m] \) with degree \( d \), and positive integers \( k \), it holds that

\[
e_k(\partial)TP(1) = \frac{(d - k + 1)(m - k + 1)}{dm} e_{k-1}(\partial)P(1) + \frac{d - k}{d} e_k(\partial)P(1). \quad (86)
\]

Proof. For brevity, denote by \( \partial^m x^\alpha \) the derivative \( \partial^n \partial x_{\alpha_1} \cdots \partial x_{\alpha_n} \), where \( \alpha = \{\alpha_1, \ldots, \alpha_n\} \subseteq [m] \). With some algebraic manipulation, and applying Proposition 2.11,

\[
e_k(\partial)QDP(1) = \sum_{\alpha \subseteq [m]} \frac{\partial^{i\alpha} QDP(1)}{\partial x_{\alpha}}, \quad (87)
\]

\[
= \sum_{\alpha \subseteq [m]} \left[ \sum_{j \in \alpha} \frac{\partial Q \partial^{i\alpha \setminus \{j\}} DP}{\partial x_j} \right]_{1} + \sum_{\alpha \subseteq [m]} \frac{\partial^{i\alpha} P(1)}{\partial x_{\alpha}}, \quad (88)
\]

\[
= \frac{m - k + 1}{m} e_{k-1}(\partial)DP(1) + e_k(\partial)DP(1), \quad (89)
\]

\[
= \frac{m - k + 1}{m} De_{k-1}(\partial)P(1) + De_k(\partial)P(1), \quad (90)
\]

\[
= \frac{(m - k + 1)(d - k + 1)}{m} e_{k-1}(\partial)P(1) + (d - k) e_k(\partial)P(1). \quad (91)
\]

Divide by \( d \) and we are done. \( \square \)

Theorem 3.6 immediately follows for polynomials satisfying (85) for each \( k \): Noting that \( e_0(\partial)P(1) = P(1) = 1 \), induction gives

\[
e_k(\partial)P(1) \geq \prod_{i=1}^{k} \frac{(m - i + 1)(d - i + 1)}{im}, \quad (92)
\]

\[
= \frac{1}{k!m^k} (m)_k (d)_k, \quad (93)
\]

\[
= k! \binom{m}{k} \binom{d}{k} m^{-k}. \quad (94)
\]
We will call a polynomial \( P = \sum_\alpha a(\alpha) x^\alpha \in \mathbb{R}[x_1, \ldots, x_m] \) symmetric if, for any permutation \( \pi \) on \([m]\), \( a(\alpha) = a(\pi \alpha) \) for all \( \alpha \in \mathbb{N}^m \).

While (85) does not hold for all stable, doubly stochastic polynomials \( P \), it does for \( k = d = m \) if \( P \) is symmetric.

**Proposition 4.8.** Let \( P = \sum_\alpha a(\alpha) x^\alpha \in \mathbb{R}[x_1, \ldots, x_m] \) be a stable, symmetric polynomial with non-negative coefficients and degree \( m \). Then \( e_m(\partial)P(1) \geq e_m(\partial)TP(1) \).

The proof makes use of the following lemma:

**Lemma 4.9.** Let \( P \) be as in Proposition 4.8. Then \( a(1) \geq 2a(1 + e_i - e_j) \).

**Proof.** By symmetry it is sufficient to give a proof for \((i, j) = (1, 2)\). We have

\[
\left[ \frac{\partial^{m-2}P}{\partial x_3 \cdots \partial x_m} \right]_{x_3 = \cdots = x_m = 0} = a(1)x_1x_2 + a(1 + e_1 - e_2)x_1^2 + a(1 - e_1 + e_2)x_2^2,
\]

a stable polynomial by Proposition 2.5 and hence the discriminant is nonnegative,

\[
a(1)^2 \geq 4a(1 + e_1 - e_2)a(1 - e_1 + e_2).
\]

By symmetry, the coefficients in the LHS are identical and hence

\[
a(1) \geq 2a(1 + e_1 - e_2).
\]

\( \square \)

**Proof of Proposition 4.8.** Consider the polynomial \( \hat{P} = \sum_\alpha \hat{a}(\alpha) x^\alpha = TP \). We intend to show that \( \hat{a}(1) \leq a(1) \).

\[
\hat{a}(1) = \left[ \frac{x_1 + \cdots + x_m}{m^2} \sum_{j=1}^m \frac{\partial}{\partial x_j} \sum_\alpha a(\alpha) x^\alpha \right]_{x^1},
\]

(98)

\[
= \left[ \frac{1}{m^2} \sum_{i,j=1}^m \sum_\alpha a(\alpha) x_i \frac{\partial x^\alpha}{\partial x_j} \right]_{x^1},
\]

(99)

\[
= \frac{1}{m^2} \sum_{i,j=1}^m \sum_\alpha a(\alpha) \alpha_j \text{ if } e_i + \alpha - e_j = 1,
\]

(100)

\[
= \frac{1}{m^2} \sum_{i,j=1}^m \sum_\alpha \left[ 1 - e_i + e_j \right] a(1 - e_i + e_j),
\]

(101)

\[
= \frac{1}{m^2} \left( m \cdot a(1) + \sum_{i,j \in [m]} 2 \cdot a(1 - e_i + e_j) \right),
\]

(102)

\[
\leq \frac{1}{m^2} \left( m \cdot a(1) + m(m-1) \cdot a(1) \right),
\]

(103)

\[
= a(1),
\]

(104)
where we used Lemma 4.9 in the last inequality.

Gurvits’ theorem can be used to prove a special case of $m \leq 3$.

**Proposition 4.10.** Let $P = \sum a(\alpha)x^\alpha \in \mathbb{R}[x_1, \ldots, x_m]$ be a stable, doubly stochastic polynomial with non-negative coefficients and degree $m$. If $m \leq 3$ and $P$ is quadratic in the individual variables, then $e_m(\partial)P(\mathbb{1}) \geq e_m(\partial)TP(\mathbb{1})$.

**Proof.** Following the proof of Claim 4.8, it is sufficient to prove that

$$\sum_{i,j \in [m] \atop i \neq j} a(\mathbb{1} + e_i - e_j) \leq \left(\frac{m}{2}\right) a(\mathbb{1}). \quad (105)$$

We can apply Theorem 3.1 when each variable in $P$ individually has degree 2 and hence $e_1 = 1$, $e_i = 2$ for $i = 2 \ldots m$. The capacity of $P$ is 1 and hence

$$a(\mathbb{1}) \geq 2^{-m+1}, \quad (106)$$

$$= 2^{-m+1} \cdot P(\mathbb{1}), \quad (107)$$

$$= 2^{-m+1} \cdot \sum_{|\alpha| = m \atop |\alpha|_\infty \leq 2} a(\alpha), \quad (108)$$

$$\iff \quad a(\mathbb{1}) \left(2^{m-1} - 1\right) \geq \sum_{|\alpha| = m \atop |\alpha|_\infty = 2} a(\alpha), \quad (109)$$

$$\geq \sum_{i,j \in [m] \atop i \neq j} a(\mathbb{1} + e_i - e_j), \quad (110)$$

where $2^{m-1} - 1 = \binom{m}{2}$ for $m \leq 3$.

From the proof of Lemma 4.9, we can actually establish a tiny extension of Theorem 3.1.

**Theorem 4.11.** For $P = \sum_{|\alpha| = m} a(\alpha)x^\alpha \in \mathbb{R}[x_1, \ldots, x_m]$ stable, homogeneous of degree $m$ with non-negative coefficients and $1 \leq i, j \leq m$, $i \neq j$, it holds that

$$2\text{Cap}(P) \geq 2a(\mathbb{1}) \geq a(\mathbb{1}) + 2\sqrt{a(\mathbb{1} + e_i - e_j)a(\mathbb{1} - e_i + e_j)},$$

$$\geq \prod_{j=3}^{m} G(e_j)\text{Cap}(P), \geq 2\frac{m!}{m^m}\text{Cap}(P). \quad (111)$$
Proof. We may assume that $i = 1$ and $j = 2$. The first inequality comes from Theorem 3.1, the second from the non-negative discriminant in the proof of Lemma 4.9. For the last one, consider the proof of Theorem 3.1 but stopping after $m - 2$ steps of applying the differentiation inequality. We have

$$\text{Cap} \left( a(1 + e_1 - e_2)x_1^2 + a(1)x_1x_2 + a(1 - e_1 + e_2)x_2^2 \right) \geq \prod_{j=3}^{m} G(e_j) \text{Cap}(P),$$

where the first capacity evaluates to $a(1)+2\sum a(1+e_i-e_j)a(1-e_i+e_j)$ and the product is missing the factors $G(1) = 1, G(2) = \frac{1}{2}$.

4.5 An induced map of MCPs

The operator $T$ induces a map of MCPs, $\hat{T} : \chi_t[P] \mapsto \chi_t[TP]$. We dedicate this section to some of its properties.

Proposition 4.12. The map $\hat{T}$ is the linear map

$$\hat{T} p = \frac{1}{d} \left[ \left( t - \frac{m - d + 1}{m} \right) p' - \frac{t}{m} p'' \right].$$

Proof. To verify the formula for $\hat{T}$, we let $\hat{T} = (\alpha t + \beta) \frac{d}{dt} + (\gamma t + \delta) \frac{d^2}{dt^2}$ and look for solutions $(\alpha, \beta, \gamma, \delta)$.

For a polynomial $p = \sum_{i=0}^{d} a_i t^i$, let $[t^k]p = a_k$. By Lemma 4.7

$$(-1)^k [t^{d-k}] \chi_t[TP] = \frac{(m-k+1)(d-k+1)}{md} e_{k-1}(\partial) P(1) + \frac{d-k}{d} e_k(\partial) P(1).$$

Computing $[t^{d-k}]\hat{T}\chi_t[P]$ in terms of the $e_k(\partial) P(1)$ and comparing to the above, we get a system of linear equations in $\alpha, \beta, \gamma, \delta$,

$$\begin{align*}
\alpha(d-k) &= \frac{d-k}{d}; \\
(d-k+1) \left( (\beta + \gamma)(d-k+1) \right) &= -\frac{(d-k+1)(m-k+1)}{dm}; \\
\delta(d-k+1)(d-k+2) &= 0,
\end{align*}$$

for $k \in [m]$. The solution gives precisely the expression in (113).

Fixed points of $\hat{T}$ satisfy the equation $\hat{T} p = p$, which under the substitution $q(t) = p(t/m)$ yields the equation

$$tq'' + (1 + m - d - t)q' + dq = 0.$$
This is precisely Laguerre’s equation, whose solution is the associated Laguerre polynomial $L_d^{m-d}(t)$. From Theorem 4.3 we conclude

$$\chi_t[P_{J_d,m}] = L_d^{m-d}(mt). \quad (117)$$

Incidentally, as this polynomial is tight in the inequality Theorem 3.6, we have derived a formula for the associated Laguerre polynomials.

$$L_d^{m-d}(mt) = \sum_{k=0}^{d} t^{d-k} (-1)^k k! \binom{d}{k} \binom{m}{k} m^{-k}. \quad (118)$$
5 Graphs, matrices and stable polynomials

Throughout this chapter, we will be concerned with product polynomials generated from matrices and certain bipartite graphs.

We introduce the permanent in Section 5.1 and show how subpermanent sums of stochastic matrices describe the MCPs of their product polynomials. This immediately reduces van der Waerden’s permanent conjecture to a special case of Gurvits’ theorem.

In this context, a disproof of an old conjecture on matrix permanents provides a counterexample to the monotonicity of the resampling operator studied in 4.3, and simultaneously answers a question on majorization of MCP roots in the negative. We do this in Section 5.1.3.

In Section 5.2, we consider simple weighted graphs and show how their matching polynomials relate to MCPs induced by the weights. Using these results, we prove in Section 5.3 a general bound on the roots of product polynomial MCPs by extending a result of Godsil on matching polynomial divisibility.

5.1 Matrices and product polynomials

5.1.1 Permanents

Permanents are functionals of matrices that are closely related to determinants. The MCP of a product polynomial $P_A$ is completely determined by the subpermanents of $A$ when $A$ is stochastic. This reduces the study of MCPs of product polynomials to the familiar study of their matrices.

For sets $X, Y$ of equal cardinality, let $S(X, Y)$ be the set of bijections $X \rightarrow Y$, and define $S_m = S([m], [m])$, the set of permutations on $m$ elements. We define the permanent $\text{Per } A$ of a matrix $A \in \mathbb{R}^{m \times m}$ as the sum

$$\text{Per } A = \sum_{\pi \in S_m} \prod_{i=1}^{m} a_{\pi(i)}.$$  \hspace{1cm} (119)

Denote by $\sigma_k(A)$ the sum of all $k$-subpermanents of a matrix $A \in \mathbb{R}^{d \times m}$,

$$\sigma_k(A) = \sum_{|I|=k} \sum_{|J|=k} \text{Per } A^{I,J},$$  \hspace{1cm} (120)

where $A^{I,J}$ is the submatrix with the rows and columns of $A$ indexed by $I$ and $J$, respectively. Note that if $d = m$ then $\sigma_m(A) = \text{Per } A$.

If $A$ is stochastic then the sums $\sigma_k(A)$ are the coefficients (up to a sign) of the MCP of the product polynomial $P_A$. 

31
Proposition 5.1. For a stochastic matrix \( A \in \mathbb{R}^{d \times m} \),
\[
e_k(\partial) p_A(1) = \sigma_k(A).
\]

Proof.
\[
e_k(\partial) p_A(1) = \left[ \sum_{J \subseteq [m], |J| = k} \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} \prod_{j=1}^d \sum_{i=1}^m a_{ij} x_j \right]_{x = 1},
\]
\[
= \sum_{1 \leq j_1 < \cdots < j_k \leq m} \sum_{1 \leq i_1 < \cdots < i_k \leq d} \sum_{\pi \in S_k} \left( \prod_{s=1}^k a_{i_s j_{\pi(s)}} \right) \left( \prod_{l=1}^m a_{l \cdot 1} \right),
\]
\[
= \sum_{J \subseteq [m]} \sum_{I \subseteq [d]} \sum_{|J| = k} \sum_{|I| = k} \text{Per } A^I J,
\]
\[
= \sigma_k(A).
\]

5.1.2 Van der Waerden’s permanent conjecture

In 1926, van der Waerden asked for a lower bound on the permanent of a doubly stochastic matrix \([16]\). It has since been shown to be \( \frac{m!}{m^m} \), proven independently by Falikman \([17]\) and Egorychev \([18]\) in 1981. Many proofs have emerged since then, the most innovative and simple of which is the vast generalization by Gurvits in 2006, Theorem 3.1.

Theorem 5.2 (van der Waerden’s permanent conjecture). For a doubly stochastic \( m \times m \) matrix \( A \),
\[
\text{Per } A \geq \frac{m!}{m^m},
\]
with equality if and only if \( A = \frac{1}{m} J_{m,m} \).

Proof. From Proposition 5.1 it is clear that \( e_m(\partial) P_A(1) = \sigma_m(A) = \text{Per } A \) if \( A \in \mathbb{R}^{m \times m} \) is doubly stochastic. Hence Theorem 3.6 for \( P_A \) immediately implies Theorem 5.2. □
5.1.3 Wanless’ counterexample to the Holens-Doković conjecture

The attempts at proving van der Waerden’s permanent conjecture has lead to a considerable body of research on matrix permanents. A number of conjectures arose. See the survey by Cheon and Wanless for further references [19].

One such conjecture is the Holens-Doković conjecture, in our setting claiming that for doubly stochastic matrices \( A \in \mathbb{R}^{d \times m} \) and \( k \leq \min(d, m) \), it holds that

\[
\frac{(m - k + 1)(d - k + 1)}{km} \sigma_{k-1}(A) \leq \sigma_k(A).
\]  

(127)

The original conjecture was for \( d = m \), but it is nevertheless false. Wanless gives a counterexample in [20], mentioned in [19], which we will outline here.

Note the similarity with the implication of Proposition 4.6. Together with Propositions 4.6 and 5.1, monotonicity of the \( e_k(\partial)P(1) \) under applying the resampling operator \( T \) implies the Holens-Doković conjecture and is therefore false - the condition of Proposition 4.6 is therefore not always applicable. Given a counterexample \( A \) to the Holens-Doković conjecture, it holds for its product polynomial \( P_A \) that

\[
e_k(\partial)P_A(1) < e_k(\partial)TP_A(1).
\]  

(128)

To construct a counterexample Wanless notes that the Holens-Doković conjecture implies a statement about the ratio of (weighted) \((k-1)\)-matchings to \(k\)-matchings in a bipartite graph defined by the matrix. We will go into this connection at length in Section 5.2.

For \( d \times m \)-matrices, the case \( d = m \) of the Holens-Doković conjecture is simply the claim \( \frac{\sigma_{m-1}(A)}{\sigma_m(A)} \leq m^2 \). It is the claim that the ratio of near-perfect matchings to perfect matchings in regular \( m \times m \)-bipartite graphs is polynomially bounded. Wanless provides a family of matrices - bi-adjacency matrices of graphs - where this ratio grows exponentially in \( m \), firmly disproving the conjecture.

Based on this family, Wanless finds a \( 22 \times 22 \) counterexample. With weights \( (w_1, w_2, w_3) = \left( \frac{1}{2}, \frac{1}{3}, \frac{3}{5} \right) \), take \( A \) as follows.
One can with relative ease compute \( \sigma_{22}(A) = 295245/2^{40} \) and \( \sigma_{21}(A) = 143644347/2^{40} \). It holds that \( \frac{143644347}{295245} > 22^2 \).

This matrix also has another surprising property. The MCP \( \chi_t[P_A] \) of its product polynomial has a smallest root smaller than that of the scaled Laguerre polynomial \( L^0_{22}(mt) \). This disproves a spoken conjecture of Marcus (in personal communication with Brändén) that the roots of the scaled Laguerre polynomials \( L^m_d(mt) \) majorize those of other MCPs of polynomials of degree \( d \) in \( m \) variables. For a computation, see Appendix A

5.2 Stable polynomials from graphs

5.2.1 Definitions

A text on graph theory is not complete without a heavy initial section full of definitions.

We will consider simple graphs \( G = (V, E) \) with vertex sets \( V = V(G) \) and edge sets \( E = E(G) \). For a vertex \( u \in V \), let \( \delta(u) = \{v \in V : uv \in E\} \). \( v \) is hence a neighbor of \( u \) if and only if \( v \in \delta(u) \), which we write \( v \sim u \).

Given a set of vertices \( C \subseteq V(G) \) denote by \( G[C] \) the subgraph of \( G \) induced on \( C \).

Let \( G \) be bipartite with vertex bipartitions \( A,B \), and \( w : A \times B \to \mathbb{R} \) a non-negative function on pairs of vertices of \( G \). We write \( w(u,v) = w(uw) = w_{uv} = w_{vu} = w(vu) \) interchangeably for vertices \( u \in A, v \in B \), and require that \( w_{uv} = 0 \) if and only if \( uv \notin E \).
The pair \( (G, w) \) is a weighted graph. We may write only \( G \) if there is no ambiguity about the weights.

Given the weight function \( w \) we define the bi-adjacency matrix \( A_w \in \mathbb{R}^{[A] \times [B]} \) by fixing an arbitrary enumeration \( A = \{u_1, \ldots, u_{|A|}\} \) and \( B = \{v_1, \ldots, v_{|B|}\} \) and letting the \( ij \)th entry of \( A_w \) be \( w_{u_i,v_j} \).
The weighted graph

We define the $m$-biregular graph $(G, w)$ if there are non-negative integers $c, d$ such that $|\delta(u)| = c$ for all $u \in A$ and $|\delta(v)| = d$ for all $v \in B$.

The weighted graph $(G, w)$ is $(c, d)$-biregularly weighted if there are non-negative real numbers $c, d$ such that for every $u \in A$, $\sum_{v \in B} w_{uv} = c$ and for every $v \in B$, $\sum_{u \in A} w_{uv} = d$.

Define $M_k(G)$ to be the set of $k$-matchings of $G$. For a $k$-matching $M \in M_k(G)$ of a weighted graph $(G, w)$ define its weight $w(M) = \prod_{e \in M} w_e$ and let the $k$-matching number be $m_k(G, w) = \sum_{M \in M_k(G)} w(M)$. We may omit the weights if there is no ambiguity.

We define the matching polynomial of $(G, w)$ on $n$ vertices to be

$$\mu_c[G] = \sum_{k=0}^{n/2} t^{n-2k}(-1)^k m_k(G). \tag{130}$$

### 5.2.2 Matching polynomials of bipartite graphs are basically MCPs

**Proposition 5.3.** For a weighted bipartite graph $(G, w)$ with bi-adjacency matrix $A_w$, it holds that

$$m_k(G) = \sigma_k(A_w). \tag{131}$$

**Proof.** Let $V(G)$ have the bipartition $\{A, B\}$ with $A = \{a_1, \ldots, a_d\}$ and $B = \{b_1, \ldots, b_m\}$. Fix subsets $I \subset A, J \subset B$ of size $k$. A $k$-matching on $G$ will use $k$ vertices from each of the sets $A$ and $B$, corresponding to precisely one choice of $I$ and $J$. Each such matching $M$ corresponds injectively to a bijection $\pi : I \to J$, with elements $i\pi(i) \in M$ for each $i \in I$, and has the weight $w(M) = \prod_{uv \in M} w_{uv} = \prod_{i \in I} w_{i\pi(i)}$. Any bijection $\pi : I \to J$ not corresponding to a matching has weight zero: one of the pairs $i\pi(i)$ is not an edge and hence $w_{i\pi(i)} = 0$. We have

$$m_k(G) = \sum_{M \in M_k(G)} w(M), \tag{132}$$

$$= \sum_{I \subseteq [d]} \sum_{J \subseteq [m]} \sum_{M \in M_k(G) \{|i| = \{a_i \in I \cup \{b_j \in J\}\}}} w(M), \tag{133}$$

$$= \sum_{I \subseteq [d]} \sum_{J \subseteq [m]} \sum_{\pi \in S(I, J) \forall i \in I} \prod_{i \in I} w_{i\pi(i)}, \tag{134}$$

$$= \sum_{I \subseteq [d]} \sum_{J \subseteq [m]} \text{Per} A_w^{I, J}, \tag{135}$$

$$= \sigma_k(A_w). \tag{136}$$

$\square$
Together with Proposition 5.1 this allows us to study the MCP of a product polynomial $P_A$, $A$ stochastic, by studying the weighted graph whose bi-adjacency matrix is $A$. For a common alternative definition of the matching polynomial, $\mu_t[G,w]$ and $\chi_t[P_{A_w}]$ actually coincide. See for example $R(G;x)$ in Heilmann and Liebs famous paper [8].

**Corollary 5.4.** Let $G$ be a weighted bipartite graph with vertex sets $A, B$ of sizes $|A| = d \leq |B| = m$. If its bi-adjacency matrix $A_w$ is stochastic then

$$\sqrt{t}^{d-m} \cdot \mu_{\sqrt{t}}[G] = \chi_t[P_{A_w}].$$

(137)

**Proof.** From Propositions 5.3 and 5.1 it follows that $m_k(G) = e_k(\partial)P_{A_w}$. We have

$$\sqrt{t}^{d-m} \cdot \mu_{\sqrt{t}}[G] = \sqrt{t}^{d-m} \cdot \sum_{k=0}^{d} \sqrt{t}^{d+m-2k}(-1)^km_k(G),
= t^{\frac{d-m}{2}} \cdot \sum_{k=0}^{d} t^\frac{d-m}{2}(-1)^km_k(G),
= \sum_{k=0}^{d} t^{d-k}(-1)^km_k(G),
= \chi_t[P_{A_w}].$$

(138, 139, 140, 141)

5.3 Root bounds on graph MCPs

Let $A$ be a stochastic matrix. By Corollary 5.4 a bound $\lambda$ on the roots of a weighted bipartite graph with bi-adjacency matrix $A$ implies a bound $\lambda^2$ on the roots of $\chi_t[P_A]$. If $A$ is also doubly stochastic then the induced weighted graph $G$ is biregularly weighted and by extending classical results on matching polynomials of biregular graphs to biregularly weighted ones we obtain a general bound on the roots of $P_A$.

Sections 5.3.1 - 5.3.2 give an exposition on matching polynomials mirroring that of the first few sections of Chapter 6 in Godsil’s combinatorics textbook [21], but for integer-weighted graphs. In Section 5.3.3 we conclude bounds on the roots of matching polynomials.

5.3.1 Some properties of matching polynomials

Throughout this section, we consider weighted graphs $G = (V,E)$ on $|V| = n$ vertices with non-negative weights $w : E \to \mathbb{R}$, where $w(uv) = 0$ if $uv \notin E$.

The following is an easy result which we will use without reference many times.
Proposition 5.5. For graphs $G, H$ on different vertex sets,

$$\mu_t[G \cup H] = \mu_t[G] \cdot \mu_t[H].$$  \hfill (142)

**Proof.** There’s a one-to-one correspondence between matchings of $G \cup H$ and matchings $M, M'$ of $G, H$. $M \cup M'$ is a $(|M| + |M'|)$-matching of $G \cup H$. Conversely, a matching $M''$ of $G \cup H$ can be uniquely decomposed into two matchings $M'' \cap E(G)$ and $M'' \cap E(H)$ of $G$ and $H$ respectively. Hence

$$m_k(G \cup H) = \sum_{i=0}^{k} m_i(G) \cdot m_{k-i}(H).$$  \hfill (143)

It follows that

$$\mu_t[G \cup H] = \sum_{k=0}^{n/2} t^{|G|+|H|}(-1)^k \sum_{j+j'=k} m_j(G)m_j'(H),$$  \hfill (144)

$$= \sum_{k=0}^{n/2} \sum_{j+j'=k} (t^{|G|}(-1)^j m_j(G)) \cdot (t^{|H|}-2j'(-1)^j m_j'(H)),$$  \hfill (145)

$$= \mu_t[G] \cdot \mu_t[H].$$  \hfill (146)

□

Lemma 5.6. For any vertex $u \in V(G)$ it holds that

$$\mu_t[G] = t \cdot \mu_t[G - u] - \sum_{v \sim u} w_{uv} \mu_t[G - u - v].$$  \hfill (147)

**Proof.** We will use a recursion for the $k$-matching value $m_k(G)$. The value of matchings not covering a vertex $u$ is $m_k(G - u)$. The value of matchings containing the edge $uv$ is $w_{uv} \cdot m_{k-1}(G - u - v)$. Hence

$$m_k(G) = m_k(G - u) + \sum_{v \sim u} w_{uv} m_{k-1}(G - u - v).$$  \hfill (148)

We use this to rewrite the terms of $\mu_t[G]$,

$$t^{n-2k}(-1)^km_k(G) = t \cdot t^{n-2k}(-1)^k m_k(G - u)$$

$$- t^{n-2(k-1)}(-1)^{k-1} \sum_{v \sim u} w_{uv} m_{k-1}(G - u - v),$$  \hfill (149)

37
and so

$$\mu_t[G] = \sum_{k=0}^{n/2} t^{n-2k}(-1)^km_k(G),$$

(150)

$$= t \cdot \sum_{k=0}^{n/2} t^{n-1-2k}(-1)^km_k(G-u)$$

$$- \sum_{u \sim u} w_{uv} \sum_{k=1}^{(n-2)/2+1} t^{n-2-2(k-1)}(-1)^{k-1}m_{k-1}(G-u-v),$$

(151)

$$= t \cdot \mu_t[G-u] - \sum_{u \sim u} w_{uv} \cdot \mu_t[G-u-v].$$

(152)

5.3.2 Real-rootedness and divisibility of matching polynomials

Let $G = (V, E)$ be an unweighted multigraph graph with $w_{uv}$ edges between the vertices $u, v \in V(G)$. The path-tree $T_u(G)$ of $G$ rooted at $u \in V(G)$ has as vertices the paths in $G$ starting at $u$. Two paths are adjacent if one is a maximal proper subpath of the other.

If $u_0u_1 \cdots u_k$ is a sequence of adjacent vertices in $G$ then its path-tree has vertices identified by sequences $u_0i_1 u_1i_2 u_2 \cdots i_k u_k$ for integers $i_j \in w_{u_{j-1}u_j}$, $j \in [k]$. The paths are dependent on which edge between adjacent vertices is used.

Let $G = (V, E)$ be a simple, weighted graph with integer weights. We define the path-tree $T_u(G)$ of $G$ rooted at $u \in V(G)$ to be the path tree of the multigraph $(V, E)$ with $w_{uv}$ edges between vertices $u$ and $v$. See Figure 2 for an example.

The main theorem of this section relates the matching polynomials of integer-weighted graphs to those of their path-trees. The matching polynomials of trees are their characteristic polynomials, which are well-known to be real-rooted.

Theorem 5.7. For a weighted graph $G$ and vertex $u \in V(G)$, let $C_u$ be the vertices of the component of $G$ containing $u$. If the weights are positive integers, then $\mu_t[G[C_u]]$ divides $\mu_t[T_u(G)]$.

Godsil’s proof is inductive on the number of vertices, based on the following lemma.

Lemma 5.8. For every integer-weighted graph $G$ and vertex $u \in V(G)$,

$$\frac{\mu_t[G-u]}{\mu_t[G]} = \frac{\mu_t[T_u(G)-u]}{\mu_t[T_u(G)]}.$$
Proof. By induction on the number of vertices.

If $|V| = 0$, the lemma holds vacuously.

If $|V| = 1$ then $\mu_t[G] = t = \mu_t[T_u(G)]$, $\mu_t[G - u] = 1 = \mu_t[T_u(G) - u]$ and the lemma follows.

If $|V| = 2$, then either $G = (\{u, v\}, \emptyset)$ with trivial matching polynomials, or $G = (\{u, v\}, \{uv\})$ with edge weight $w_{uv} = w$. Then the path-tree is a star graph with matching polynomial $\mu_t[T_u(G)] = t^{w-1}(t^2 - w) = t^{w-1}\mu_t[G]$. Deleting $u$ in either graph leaves monomial matching polynomials in degrees such that the lemma holds.

From now on, assume $|V| \geq 3$. Directly applying Lemma 5.6

\[
\frac{\mu_t[G]}{\mu_t[G - u]} = \frac{t\mu_t[G - u] - \sum_{v \sim u} w_{uv}\mu_t[G - u - v]}{\mu_t[G - u]}, 
\]

(154)

\[
= t - \sum_{v \sim u} \frac{w_{uv}\mu_t[G - u - v]}{\mu_t[G - u]}, 
\]

(155)

\[
= t - \sum_{v \sim u} \frac{\mu_t[T_v(G - u) - v]}{\mu_t[T_v(G - u)]}, 
\]

(156)

where the last step follows from the induction hypothesis.

We have some acrobatics ahead to recover a quotient of matching polynomials. We can decompose

\[
T_u(G) - u = \bigcup_{v \sim u} [T_v(G - u)]^{w_{uv}}, 
\]

(157)
and hence
\[
\mu_t[T_u(G) - u] = \prod_{u \sim u} \mu_t[T_v(G - u)]^{w_{uv}}.
\] (158)

Therefore, for \( v \sim u \), it holds that
\[
\mu_t[T_u(G) - u] = \left( \prod_{s \sim u \atop s \neq u} \mu_t[T_s(G - u)]^{w_{us}} \right) \cdot \mu_t[T_v(G - u)]^{w_{uv}},
\] (159)

and finally for \( k \in [w_{uv}] \),
\[
\mu_t[T_u(G) - u - ukv] = \left( \prod_{s \sim u \atop s \neq u} \mu_t[T_s(G - u)]^{w_{us}} \right) \cdot \mu_t[T_v(G - u - v)]^{w_{uv}} \cdot \mu_t[T_v(G - u)]^{w_{uv} - 1}.
\] (160)

Taking their quotient,
\[
\frac{\mu_t[T_u(G) - u - ukv]}{\mu_t[T_u(G) - u]} = \prod_{s \sim u \atop s \neq u} \frac{\mu_t[T_s(G - u)]^{w_{us}}}{\mu_t[T_v(G - u)]},
\] (161)
\[
= \frac{\mu_t[T_v(G - u - v)]}{\mu_t[T_v(G - u)]},
\] (162)
\[
= \frac{\mu_t[T_u(G) - u - ukv]}{\mu_t[T_u(G) - u]}. \tag{163}
\]

We can now continue rewriting the quotient (156), applying Lemma 5.8 to the path-tree this time.
\[
\frac{\mu_t[G]}{\mu_t[G - u]} = t - \sum_{v \sim u} w_{uv} \frac{\mu_t[T_v(G - u)]}{\mu_t[T_v(G - u)]},
\] (164)
\[
= t - \sum_{v \sim u} w_{uv} \frac{\mu_t[T_u(G) - u - ukv]}{\mu_t[T_u(G) - u]},
\] (165)
\[
= \frac{\mu_t[T_u(G)]}{\mu_t[T_u(G) - u]}. \tag{166}
\]

Take reciprocals and we are done. \( \square \)

**Proof of Theorem 5.7.** If \(|V| \leq 2\), the matching polynomials computed in the proof of Lemma 5.8 are easily verified to satisfy the theorem.

We may assume inductively that \( G \) is connected, as otherwise we simply apply the theorem to whatever component of \( G \) contains \( u \).
From now on, let $G$ be connected and $|V| \geq 3$. Then $\mu_t[G[C_u]] = \mu_t[G]$.

By Lemma 5.8

$$\mu_t[T_u(G)] = \nu_t[G] \cdot \frac{\mu_t[T_u(G) - u]}{\mu_t[G - u]}.$$  \hfill (167)

It suffices to prove that $\mu_t[G - u]$ divides $\mu_t[T_u(G) - u]$.

For any graph $H$, let $\mathcal{C}(H)$ be the partition of $V(H)$ into components of $H$. Fix an arbitrary function $U : 2^V \to V$ which endows each set of vertices of $G$ with a representative. Since $G$ is connected, it holds for any component $C \in \mathcal{C}(G - u)$ that $C \cap \delta(u) \neq \emptyset$.

We have

$$\mu_t[G - u] = \prod_{C \in \mathcal{C}(G - u)} \mu_t[(G - u)[C]].$$  \hfill (168)

By induction, this polynomial divides the product of path-trees,

$$\prod_{C \in \mathcal{C}(G - u)} \mu_t[T_{U(C \cap \delta(u))}(G - u)],$$  \hfill (169)

which, if each weight is positive, divides the product

$$\prod_{v \sim u} \mu_t[T_v(G - u)]^{w_{uv}} = \nu_t[T_u(G) - u].$$  \hfill (170)

The theorem immediately follows. \hfill \blacksquare

The significance of this theorem lies in how the matching polynomials of trees are their characteristic polynomials. It follows that for a tree $T$, $\mu_x[T]$ is the characteristic polynomial of its real, symmetric adjacency matrix and hence it is real-rooted.

**Proposition 5.9.** Let $T = (V, E)$ be an unweighted tree whose adjacency matrix $A \in \mathbb{R}^{V \times V}$ has entries

$$A_{uv} = \begin{cases} 1 & \text{if } uv \in E, \\ 0 & \text{otherwise}. \end{cases}$$  \hfill (171)

It holds that

$$\mu_x[T] = \det (xI - A_T).$$  \hfill (172)

Theorem 5.7 and Proposition 5.9 immediately imply real-rootedness of matching polynomials of any non-negatively integer-weighted graph.

**Corollary 5.10.** Let $(G, w)$ be a weighted graph. If $w$ is integer-valued, then $\mu_t[G]$ has real roots.
Proof of Proposition 5.9 Expanding the determinant,

\[
\det (xI - AT) = \sum_{\pi \in S_n(V)} \text{sgn}(\pi) \prod_{u \in V} (xI - AT)_{u \pi(u)},
\]

and noting that \(\mu_k = \sum_{M \in M_k(T)} w(M)\), we get

\[
\sum_{k=0}^{n} x^{n-2k}(-1)^k \sum_{\pi \in S_n(V) \atop \pi(\pi) = \text{id}, |\pi| = k} \prod_{u \pi(u) \neq u} (AT)_{u \pi(u)} = \sum_{k=0}^{n} x^{n-2k}(-1)^k \sum_{M \in M_k(T)} w(M),
\]

\[
= \mu_x[T].
\]

5.3.3 From divisiblity to root bounds

It is a well-known result that the eigenvalues of a tree whose vertices have degrees alternately bounded by integers \(c\) and \(d\) have a spectrum bounded in terms of \(c\) and \(d\). It follows from Corollary 4.5 in [10] together with Corollary 4.5 in [9].

Theorem 5.11 (Godsil and Mohar, [10], [9]). Let \(T = (V, E)\) be a tree. Fix \(u \in V\) and let \(h(v)\) be the distance from \(v \in V\) to \(u\) in \(T\). If there are integers \(c\) and \(d\) such that \(|d(v)| \leq c\) whenever \(h(v)\) is even and \(|d(v)| \leq d\) whenever \(h(v)\) is odd, then the spectrum of \(T\) is bounded by \(\sqrt{c - 1} + \sqrt{d - 1}\).

Corollary 5.12. Let \((G, w)\) be a \((c, d)\)-biregularly weighted graph. If \(w\) is positive and integer-valued, then the roots of \(\mu_u[G]\) are bounded by \(\sqrt{c - 1} + \sqrt{d - 1}\).

Proof. If \(G\) is disconnected then the reasoning applies separately to each of its components. Assume \(G\) is connected and let \(u \in V\).

By Theorem 5.7, \(\mu_u[G]\) is a divisor of \(\mu_u[T_u(G)]\). Hence all roots of \(\mu_u[G]\) are bounded by the largest root of \(\mu_u[T_u(G)]\). By construction the path-tree \(T_u(G)\) satisfies the conditions of Theorem 5.11 with bounds \(c, d\) and hence its roots are bounded by \(\sqrt{c - 1} + \sqrt{d - 1}\). \(\Box\)
We can now by a limit argument extend this result to arbitrary biregularly weighted graphs.

**Theorem 5.13.** Let \((G, w)\) be a \((c, d)\)-biregularly weighted graph. The roots of \(\mu_t[G]\) are bounded by \(\sqrt{c} + \sqrt{d}\).

**Proof.** Define the finite-dimensional vector space \(W\) of functions \(w : V^2 \to \mathbb{R}\) for which there are \(c, d \in \mathbb{R}\) such that \(\sum_{u \sim v} w(u, v) = c\) for all \(u \in V\) and \(\sum_{u \sim v} w(u, v) = d\) for all \(v \in V\).

Let \((G, w)\), \(G = (V, E)\), be a \((c, d)\)-biregularly weighted graph.

By the density of the reals there is a sequence \(\{w_n\} \subset W\) of functions such that \(\lim_{n \to \infty} w_n = w\). Since \(w\) is non-negative, the \(w_n\) can be taken to be positive on edges of \(G\). Each such function acts as \((c_n, d_n)\)-biregular weights on \(G\), where \(c_n \to c\) and \(d_n \to d\) as \(n \to \infty\). There are positive integer-valued functions \(\hat{w}_n\) and positive integers \(D_n\) such that \(w_n = \frac{1}{D_n} \hat{w}_n\). They act as \((D_n c_n, D_n d_n)\)-biregular weights on \(G\).

Corollary 5.12 applies to the weighted graphs \((G, \hat{w}_n)\). Hence the roots of \(\mu_t[G, \hat{w}_n]\) are bounded by \(\sqrt{D_n c_n - 1 + \sqrt{D_n d_n - 1}}\). We have by rescaling that \(\mu_t[G, D \cdot w] = D^n \cdot m_n(G, w)\) and hence

\[
\mu_t[G, w_n] = \mu_t[G, \frac{1}{D_n} \hat{w}_n],
\]

\[
= \sum_{k=0}^{\lfloor |V|/2 \rfloor} t^{\lfloor |V|/2 \rfloor - 2j} (-1)^k m_k(G, \frac{1}{D_n} \hat{w}_n),
\]

\[
= \sum_{k=0}^{\lfloor |V|/2 \rfloor} (t)^{\lfloor |V|/2 \rfloor - 2j} (-1)^k D_n^{-k} m_k(G, \hat{w}_n),
\]

\[
= \sqrt{D_n}^{\lfloor |V|/2 \rfloor} \mu_t[D_n^{-1} \hat{w}_n].
\]

We conclude that the roots of \(\mu_t[G, w_n]\) are bounded by \(\sqrt{D_n c_n - 1 + \sqrt{D_n d_n - 1}} \leq \sqrt{c} + \sqrt{d}\).

As we take the limit \(n \to \infty\), we have that \(w_n \to w\), \(c_n \to c\), \(d_n \to d\), \(\mu_t[G, w_n] \to \mu_t[G, w]\) and hence the roots of \(\mu_t[G, w]\) are bounded by \(\lim_{n \to \infty} \sqrt{c} + \sqrt{d} = \sqrt{c} + \sqrt{d}\).

We can finally conclude a general bound on the roots MCPs of product polynomials of doubly stochastic matrices.

**Corollary 5.14.** Let \(A\) be a doubly stochastic \(d \times m\) matrix. The roots of \(\chi_t[P_A]\) are bounded by \(\left(1 + \sqrt{\frac{d}{m}}\right)^2\).
Proof. Let \((G, w)\) be the \((1, \frac{d}{m})\)-biregularly weighted graph defined by \(G = K_{d,m}\), the complete bipartite graph on \(d\) and \(m\) vertices, with weight matrix \(A_w = A\). Let \(\lambda\) be a non-zero root of \(\chi_t[P_A]\). By Proposition 4.1, \(\lambda\) is positive. By Corollary 5.4, \(\sqrt{\lambda}\) is a root of \(\mu_t[G, w]\). Hence, by Corollary 5.13, \(\lambda\) is bounded by \(\left(1 + \sqrt{\frac{d}{m}}\right)^2\). \(\square\)
6 Future work

Improving Gurvits’ theorem for individually degree-bounded polynomials from the combinatorial perspective could give exact bounds on matching counts, related to the Lower matching conjecture for which most current results are asymptotic, see the article by Csikvári [22].

Gurvits’ theorem is tight when polynomials are quadratic in each variable (Section 4.2) and when no individual bound is assumed (Section 3.3). Computer-aided investigations by the author hint that doubly stochastic, stable polynomials in $m$ variables with degree at most $m$ have multilinear coefficient bounded when $m = 4$ and the polynomial is cubic in each variable by about $\frac{1}{9}$, rather than $\frac{8}{81}$ as given by Theorem 3.1. For $m$ up to 6 there seems to be better bounds for all individual degree-bounds from 3 to $m - 1$.

No ground has been gained on determining if scaled associated Laguerre polynomials $L_{m-d}^d(mt)$ have the largest roots among MCPs of $d$-degree doubly stochastic stable polynomials in $\mathbb{R}[x_1, \ldots, x_m]$. Proving this would render both the bound of Corollary 5.14 and the barrier argument in [6] obsolete.

Throughout the computer-aided investigations of the author, admittedly on low-dimensional polynomials, the largest root of MCPs has increased when applying the operator operator $\hat{T}$. If this holds true in general, a proof of this would by Theorem 4.3 imply the Laguerre polynomial bound.

An alternate path to Corollary 5.14 is through bounding the spectrum of infinite $(c, d)$-biregularly weighted trees. The results of Section 5.3.2 have formulations in terms of arbitrary weight functions, not just integer-valued, if one defines the path-tree to retain weights. If the results of Godsil and Mohar (Theorem 5.11) have corresponding generalizations then one achieves the same bound directly without the limit argument. More importantly, one generalizes known bounds to spectra of infinite weighted graphs.

Finally, it is not known if there are MCPs that are not generated by a product polynomial. If there are no such MCPs then Corollary 5.14 circumvents the barrier argument in [6].
## 7 Index of notation

### Constants

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>$(d)_k$</td>
<td>11</td>
</tr>
<tr>
<td>$\bar{e}_i$</td>
<td>17</td>
</tr>
<tr>
<td>$S(n,k)$</td>
<td>24</td>
</tr>
</tbody>
</table>

### Operators

<table>
<thead>
<tr>
<th>Operator</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cap</td>
<td>14</td>
</tr>
<tr>
<td>$D$</td>
<td>11</td>
</tr>
<tr>
<td>$\partial_n x^\alpha$</td>
<td>26</td>
</tr>
<tr>
<td>$e_k(\partial)$</td>
<td>8</td>
</tr>
<tr>
<td>Per</td>
<td>31</td>
</tr>
<tr>
<td>$\sigma_k$</td>
<td>31</td>
</tr>
<tr>
<td>$T$</td>
<td>22</td>
</tr>
</tbody>
</table>

### Polynomials, matrices and graphs

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{I,J}$</td>
<td>31</td>
</tr>
<tr>
<td>$A_w$</td>
<td>34</td>
</tr>
<tr>
<td>$E(G)$</td>
<td>34</td>
</tr>
<tr>
<td>$e_k(x)$</td>
<td>8</td>
</tr>
<tr>
<td>$G$</td>
<td>34</td>
</tr>
<tr>
<td>$G[C]$</td>
<td>34</td>
</tr>
<tr>
<td>$J_{d,m}$</td>
<td>16</td>
</tr>
<tr>
<td>$M_k(G)$</td>
<td>35</td>
</tr>
<tr>
<td>$m_k(G,w)$</td>
<td>35</td>
</tr>
<tr>
<td>$\mu_G$</td>
<td>35</td>
</tr>
<tr>
<td>$P_A$</td>
<td>4</td>
</tr>
<tr>
<td>$Q$</td>
<td>22</td>
</tr>
<tr>
<td>$T_{u(G)}$</td>
<td>38</td>
</tr>
<tr>
<td>$\chi_P$</td>
<td>20</td>
</tr>
<tr>
<td>$V(G)$</td>
<td>34</td>
</tr>
<tr>
<td>$w$</td>
<td>34</td>
</tr>
</tbody>
</table>

### Sets

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>4</td>
</tr>
<tr>
<td>$[d]$</td>
<td>4</td>
</tr>
<tr>
<td>$\delta(u)$</td>
<td>34</td>
</tr>
<tr>
<td>$H_+$</td>
<td>4</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>4</td>
</tr>
<tr>
<td>$S(X,Y)$</td>
<td>31</td>
</tr>
<tr>
<td>$S_m$</td>
<td>31</td>
</tr>
</tbody>
</table>
References


A Appendix: Wanless’ counterexample MCP

This appendix contains an outline of the computation of the MCP of Wanless’ counterexample from Section 5.1.3.

For a graph $G = (V, E)$, a vertex $v$ and an edge $e$, define $G + v$ to be the graph $(V \cup \{v\}, E)$ and $G + e$ to be $(V, E \cup \{e\})$.

Define a graph $H_s$ by $V(H_s) = \{s_i : s \in [6]\}$ with edges

$$E(H_s) = \{s_1s_2, s_1s_3, s_2s_4, s_2s_5, s_3s_4, s_3s_5, s_4s_6, s_5s_6\}.$$  \hspace{1cm} (181)

Define weights by

$$w(s_i s_j) = \begin{cases} w_3 & \text{if } \{i, j\} \cap \{1, 6\} = \emptyset, \\ w_2 & \text{otherwise.} \end{cases}$$  \hspace{1cm} (182)

Figure 3: The constructions for computing matchings of Wanless’ counterexample.

Define graphs $H_{s,t}^n = (\{s, t\}, \{st\})$ with weight $w(st) = w_1$, and for $n > 1$,

$$H_{s,t}^n = H_{s,t}^{n-1} \cup H_n + t + u_6t,$$  \hspace{1cm} (183)

with additional weight $w(u_6t) = w_1$.

Define $G_{n_1, n_2} = H_{s,1}^{n_1} \cup H_{s,1}^{n_2}$. Note that $G_{3,4}$ is the graph defined by Wanless’ counterexample.

**Fact A.1.** For integers $n, k \leq 3n + 1$, $a_i$ for $i \in [3]$ such that $a_1 + a_2 + a_3 = k$ and $l, l' \in \{0, 1\}$, let $M(n, k, a_1, a_2, a_3, l, l')$ be the number of $k$-matchings on $H_{s,t}^n$ with $a_i$ vertices with weight $w_i$ for $i \in [3]$ and $s$ or $t$ matched if and only if $l = 1$ or $l' = 1$ respectively.

Then

$$M(0, k, a_1, a_2, a_3, l, l') = \begin{cases} 1 & \text{if } (k, a_1, a_2, a_3, l, l') = (B, B, 0, 0, B, B) \text{ where } B \in \{0, 1\}, \\ 0 & \text{otherwise,} \end{cases}$$  \hspace{1cm} (184)
and for \( n \geq 1, \)

\[
M(n, k, a_1, a_2, a_3, l, l') = \begin{cases} 
2 \cdot M(n - 1, k - 1, l, 0, a, b - 1, c) \\
+ 2 \cdot M(n - 1, k - 1, l, 1, a, b - 1, c) \\
+ 4 \cdot M(n - 1, k - 2, l, 0, a, b - 2, c) \\
+ 4 \cdot M(n - 1, k - 3, l, 0, a, b - 2, c - 1) \\
+ 4 \cdot M(n - 1, k - 2, l, 0, a, b - 1, c - 1) \\
+ 4 \cdot M(n - 1, k - 1, l, 0, a, b - 1, c - 1) \\
+ 4 \cdot M(n - 1, k - 1, l, 1, a, b, c - 1) \\
+ 2 \cdot M(n - 1, k - 2, l, 0, a, b, c - 2) \\
+ 2 \cdot M(n - 1, k - 2, l, 1, a, b, c - 2) \\
+ 4 \cdot M(n - 1, k - 2, l, 0, a, b - 1, c - 1) \\
+ 2 \cdot M(n - 1, k - 1, l, 0, a, b - 1, c) \\
+ M(n - 1, k, l, 0, a, b, c) \\
+ M(n - 1, k, l, 1, a, b, c) \\
M(n - 1, k - 1, l, 0, a - 1, b, c) \\
+ M(n - 1, k - 1, l, 1, a - 1, b, c) \\
+ 4 \cdot M(n - 1, k - 2, l, 0, a - 1, b, c - 1) \\
+ 4 \cdot M(n - 1, k - 2, l, 1, a - 1, b, c - 1) \\
+ 2 \cdot M(n - 1, k - 3, l, 0, a - 1, b, c - 2) \\
+ 2 \cdot M(n - 1, k - 3, l, 1, a - 1, b, c - 2) \\
+ 2 \cdot M(n - 1, k - 2, l, 0, a - 1, b - 1, c) \\
+ 4 \cdot M(n - 1, k - 3, l, 0, a - 1, b - 1, c - 1) & \text{if } l' = 0, \\
\end{cases} 
\]

(185)

The number of matchings of \( G_{n_1,n_2} \) can now easily be computed recursively (with aggressive memoization) for sufficiently small values of \( n_1, n_2 \) as

\[
m_k(G_{n_1,n_2}) = \sum_{i=0}^{k} \sum_{(i,l'_1,l'_2) \in L} \left( \sum_{a_1+a_2+a_3=i} M(n_1, i, a_1, a_2, a_3, l_1, l'_1) w_1^{a_1} w_2^{a_2} w_3^{a_3} \right) \cdot \left( \sum_{a_1+a_2+a_3=k-i} M(n_2, k-i, a_1, a_2, a_3, l_1, l'_1) w_1^{a_1} w_2^{a_2} w_3^{a_3} \right), 
\]

(186)

for \( L = \{ l \in \{0,1\}^4 : l_1 \cdot l_2 \neq 1, l_3 \cdot l_4 \neq 1 \} \). It is here the case that \( w_1, w_2, w_3 \) and hence \( m_k(G_{n_1,n_2}) \) are rational and can be determined exactly.
By Propositions 5.1 and 5.3 it follows that the MCP of Wanless’ counterexample satisfies

\[2^{40} \cdot \mu_2(G_{3,4}) = 1099511627776 + 24189255811072x + 246256244883456x^2 + 1540192452214784x^3
\]
\[+ 6626856438595584x^4 + 20815424782336000x^5 + 49440127780388864x^6
\]
\[+ 9076091733052192x^7 + 130542353846894592x^8 + 14826593598203494x^9
\]
\[+ 13342841945448096x^{10} + 9510810535743072x^{11} + 53488147262651392x^{12}
\]
\[+ 23554849430486016x^{13} + 8027417270841888x^{14} + 2081476529958960x^{15}
\]
\[+ 400922025297120x^{16} + 5544201547664x^{17} + 523669900566x^{18}
\]
\[+ 31266583437x^{19} + 10336730112x^{20} + 14364347x^{21} + 295245x^{22}.
\]

(187)

By applying Sturm’s algorithm to the above polynomial one can indeed confirm that this polynomial has a smallest root less than 0.0025, as opposed to the corresponding associated Laguerre polynomial \(L_0^{22}(22 \cdot t)\).

This can be done using the Python and Mathematica code available for an undeterminable time in the repository at [23].